Relating Constrained Motion to Force Through Newton's Second Law

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Relating Constrained Motion to Force Through Newton's Second Law

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PREFACE

The reader who is viewing this work in its electronic form on a computer, that is to say in Portable Document Format, may find it helpful to know that hyperlinks are provided for entries in the table of contents, the list of tables, and the list of figures, and for numbers given to chapters, sections, subsections, equations, references, and examples; unfortunately, Institute constraints require that colors be absent from these hyperlinks.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

SUMMARY

When a mechanical system is subject to constraints its motion is in some way restricted. In accordance with Newton's second law, motion is a direct result of forces acting on a system; hence, constraint is inextricably linked to force. The presence of a constraint implies the application of particular forces needed to compel motion in accordance with the constraint; absence of a constraint implies the absence of such forces.

The objective of this thesis is to formulate a comprehensive, consistent, and concise method for identifying a set of forces needed to constrain the behavior of a mechanical system modeled as a set of particles and rigid bodies. The goal is accomplished in large part by expressing constraint equations in vector form rather than entirely in terms of scalars. A constraint equation that has been differentiated once or twice with respect to time, so that it contains the acceleration of a point or the angular acceleration of a rigid body, is said to be written at the acceleration level. Likewise, a constraint equation at the velocity level is one that has been differentiated at most once, so that it contains the velocity of a point or the angular velocity of a rigid body. The method developed here can be applied whenever constraints can be described at the acceleration level by a set of independent equations that are linear in acceleration. Hence, the range of applicability extends to servo-constraints or program constraints described at the velocity level with relationships that are nonlinear in velocity. All configuration constraints, and an important class of classical motion constraints, can be expressed at the velocity level by using equations that are linear in velocity; therefore, the associated constraint equations are linear in acceleration when written at

the acceleration level.

A review of the literature shows that current methods for determining constraint forces (also called reaction forces or noncontributing forces) lack consistency and a comprehensive scope. Some are limited to configuration constraints that do not involve prescribed position. Others include motion constraints, but only of a certain type, such as rolling. In one case, two different approaches are used to relate undetermined multipliers to forces associated with configuration and motion constraints. In other cases obtaining such relationships involves a certain amount of wasted effort or leads to results that are at odds with physical reasoning. Some methods must be applied to all constraints that are imposed even though the analyst may only be interested in a few of them. Complications in several methods are observed to arise from a predilection to work exclusively with scalars rather than vectors.

The broad scope of the method set forth herein is demonstrated by applying it to a wide range of constraints encountered in practice, including those associated with confinement of a particle in a rigid body, joints, prescribed position, constant distance between particles, impenetrability of rigid bodies, rolling, and sharp-edged blades. Two of Kepler's laws of planetary motion are regarded as constraints to show that if rules describing observations of a physical phenomenon can be couched as restrictions on position, velocity, or acceleration, then the method may be employed to deduce a force law from the rules. The method is applied also to less commonly considered servo-constraints such as the requirement that the velocities of two separate particles remain parallel, equal in magnitude, or perpendicular. Additionally, the method is used to uncover flaws in an assertion made to support an incorrect approach to dealing with nonholonomic constraint equations within the framework of variational principles.

Two new approaches are presented for deriving equations governing motion of a system subject to constraints expressed at the velocity level with equations that are nonlinear in velocity. By using partial accelerations instead of the partial velocities normally employed with Kane's method, it is possible to form dynamical equations that either do or do not contain evidence of the constraint forces, depending on the analyst's interests.

CHAPTER 1

INTRODUCTION

In this first chapter the reader is introduced to approaches in current use for dealing with constrained motion and determining forces of constraint. Relationships between Lagrange multipliers or undetermined multipliers and constraint forces are examined closely. Reasons for dissatisfaction with the way in which these relationships are obtained, and with several other aspects of the present state of the art, are spelled out in detail. This is followed by a review of recent suggestions for using vectors rather than scalars in describing constraints; as it turns out, this practice can lead to a method that does not suffer from the deficiencies observed in the approaches used to date. The chapter concludes with an overview of the remainder of the thesis.

1.1 General Approaches for Dealing With Constraints

Equations of motion for systems made up of rigid bodies subject to configuration and motion constraints generally are formulated in one of three ways. The constraint equations may be embedded into dynamical equations of motion, the number of which is equal to the degrees of freedom p possessed by the system. Such equations involve p independent generalized accelerations, and the forces that constrain the system are not in evidence. Alternatively, n generalized accelerations, of which only p are independent, appear in n dynamical equations of motion to which are adjoined m independent equations of constraint. The number of degrees of freedom p is equal to $n-m$, and the constraint equations are adjoined by means of m quantities referred to as Lagrange multipliers or undetermined multipliers that are related to the constraint forces. In this approach n dynamical equations and m constraint equations are solved for n generalized accelerations and m multipliers. A third approach exists in which n dynamical equations involving n generalized accelerations, not all independent, are formed without bringing multipliers into the picture.

The configuration of a system in a reference frame is described uniquely by a set of generalized coordinates that are by definition independent of one another, and thus minimum in number. Therefore, holonomic constraint equations are considered to be embedded in dynamical equations whenever one works with a set of generalized coordinates. The methods of Newton-Euler, Lagrange, and Kane, when applied to holonomic systems, are thus well known examples of embedding the constraint equations. The number of generalized coordinates is typically denoted by n, and for holonomic systems this is equal to the number of degrees of freedom, $p = n$. In Lagrange's method (Ref. [33], pp. 256, 257; Ref. [57], p. 75; Ref. [82], p. 37) a generalized acceleration is simply the second derivative of a generalized coordinate with respect to the time t. The time derivative of a motion variable plays the part of a generalized acceleration in Kane's method (Ref. [44], p. 159); a motion variable, also called a generalized speed, is itself a linear combination of the time derivatives of the generalized coordinates (Ref. [44], p. 40). Kane's method applied to a system subject to m motion constraints (Ref. [44], p. 43) furnishes another example of embedding the constraint equations; one obtains p dynamical equations (Ref. [44], p. 158), where the number of degrees of freedom of a simple nonholonomic system is $p = n - m$, expressed in terms of p independent motion variables and their time derivatives. The forces responsible for imposing constraints are introduced at an early stage in the application of the Newton-Euler method, and usually are subsequently eliminated. In comparison, Lagrange's and Kane's equations for holonomic systems, and Kane's equations for simple nonholonomic systems, all offer the advantage that constraint forces need not be introduced (unless they happen to be required for use with a friction model); with Lagrange's approach such forces are referred to as nonworking whereas Kane refers to them as noncontributing. It is important to note that when these forces are of interest Kane's method contains provisions for bringing them into evidence selectively (Ref. [44], pp. 114–117) by introducing additional motion variables; one additional equation and one additional unknown scalar are produced for every additional motion variable, although the unknown may find its way into more than one equation.

Perhaps the most widely recognized example of adjoining the constraint equations is found in Lagrange's equations for nonholonomic systems (Ref. [32], pp. 38–44; Ref. [33], p. 269; Ref. [57], p. 75; Ref. [63], pp. 211–215; Ref. [82], p. 215). Nonholonomic constraint equations can be adjoined in an analogous manner to obtain a version of Kane's equations with undetermined multipliers employed by some dynamicists (for example, in Refs. [79], [1], [9], [11], and [37]). A set of m independent nonholonomic constraint equations is written as $\alpha u + \beta = 0$ where u is an $n \times 1$ array of motion variables that are not all independent, α is an $m \times n$ Jacobian, and β is an $m \times 1$ array. The configuration of the system is completely described by n generalized coordinates. In both methods the constraint equations are adjoined to the dynamical differential equations of motion governing the unconstrained system to yield $\mathcal{M}\dot{u} = f + \alpha^{\mathrm{T}}\lambda$, where $\mathcal{M}$ is an $n \times n$ positive definite mass matrix, f is an $n \times 1$ array, and λ is an $m \times 1$ array of multipliers. The $n \times 1$ array $\alpha^{\mathrm{T}}\lambda$ contains generalized constraint forces and can be denoted by F^c. The two sets of equations are $n + m$ in number, which is equal to the number of unknowns contained in $\dot{u}$ and λ. One method of obtaining the unknowns involves differentiating the nonholonomic constraint equations with respect to time to yield $\alpha\dot{u} + \gamma = 0$ and the resulting solutions, $\lambda = -(\alpha\mathcal{M}^{-1}\alpha^{\mathrm{T}})^{-1}(\gamma + \alpha\mathcal{M}^{-1}f)$ and $\dot{u} = \mathcal{M}^{-1}(f + \alpha^{\mathrm{T}}\lambda)$; however, the concomitant penalty is that approximations inherent in numerical integration of ordinary differential equations can lead to violations or drift in which the original constraint equations are unsatisfied. Drift in nonholonomic constraint equations can be avoided by leaving them in their undifferentiated form and solving the resulting index-2 differential-algebraic equations (Ref. [36], p. R-20).

Holonomic constraint equations can be likewise adjoined to $n + M$ dynamical equations in which $n + M$ coordinates are used to describe the configuration of an unconstrained system. The coordinates are related to one another by M holonomic constraint equations, and M multipliers come into play. The holonomic constraint equations differentiated with respect to time can be treated like nonholonomic constraint equations as described previously; however, the possibility of constraint drift exists. Alternatively, drift can be eliminated by working with the constraint equations in their original form and solving the associated index-3 differential-algebraic equations (Refs. [14], [15], and [50]). Recent research (Refs. [20] and [21]) shows that proper scaling can be used to surmount the numerical difficulties heretofore encountered in solving the index-3 equations, making them as easy to integrate as well-behaved ordinary differential equations. As reported in Ref. [74], non-independent reference point coordinates have been used to describe holonomic systems in multibody computer programs such as NEWEUL, ADAMS, DADS, and OMEGA. There are additional examples in the field of biomechanical modeling (such as Refs. [25] and [67]) of describing a holonomic system with a set of natural coordinates proposed in Ref. [30], which are not to be confused with coordinates of the same name used in the study of vibrations and also referred to as principal or modal coordinates. The authors of Refs. [27] and [18] both note the common practice of using a large number of coordinates (up to 6) to describe the configuration of each member of a multibody system and then accounting for joints with a large number M of constraints, as exhibited in Refs. [3], [25], [30], [64], [67], and [74]. As Blajer remarks, dealing with the large sets of equations produced by this approach is computationally arduous; furthermore, effort is wasted if the analyst is not interested in all of the associated constraint forces and torques.

The third approach to dealing with constrained systems produces as many dynamical equations as the minimal number of coordinates required to describe the configuration of the unconstrained system. In the case of a nonholonomic system described by n generalized coordinates there are n dynamical equations, whereas $n+M$ equations are obtained for a holonomic system described by $n+M$ coordinates subject to M configuration constraints. The dynamical equations so derived are free of Lagrange multipliers or any other unknowns representing the constraint forces. Because there are more dynamical equations than there are system degrees of freedom, these sets of equations are referred to variously as nonminimal, unreduced, or full order.

Equations (16) presented in Ref. [74] constitute a nonminimal set governing the motion of systems of bodies fastened together by joints whose effects are necessarily described by scleronomic holonomic constraint equations. The nonminimal equations are applicable also to systems subject to motion constraints expressed with catastatic nonholonomic equations, but the authors do not point this out. Udwadia and Kalaba appear to have been the first to develop nonminimal equations for holonomic and nonholonomic systems whose restrictions can be described with equations that are rheonomic or acatastatic in general but can be, respectively, scleronomic or catastatic as special cases. These nonminimal equations are reported in Refs. [72] and [73], where the authors propose a new fundamental principle of analytical mechanics. Chen applies several of the key techniques used by Udwadia and Kalaba to the equations of Lagrange and Maggi in Ref. [23], and Bajodah et al. continue in the same vein to obtain a new form of Kane's equations set forth in Refs. [6] and [61]. It is reported that nonminimal equations lend themselves to studies of stability, chaos, bifurcation and control system design that cannot be performed using sets of minimal equations. Arabyan and Wu note in Ref. [4] that the formulation of Udwadia and Kalaba is especially useful in a general purpose multibody computer code for dealing with systems in which constraints are intermittent or redundant, and systems undergoing motion during which the number of degrees of freedom changes. A critical step in the derivation of the nonminimal equations is observed to be the use of constraint equations that have been differentiated an appropriate number of times so that they are expressed in terms of the generalized accelerations; the result is that holonomic and nonholonomic systems are treated in a unified way. Another form of nonminimal equations free of Lagrange multipliers is presented in Ref. [63], pp. 243–245, where p Lagrange's dynamical equations are solved together with m nonholonomic constraint equations. Although Rosenberg refers to this as embedding nonholonomic equations, the dependent generalized accelerations are not eliminated so the method is not considered to represent embedding as the term is used here.

When motion of a constrained system is to be analyzed and one is uninterested in the forces that give rise to constraints, the most economical course of action is to form directly and solve equations in which the constraint equations are embedded, for example Lagrange's equations for a holonomic system, or Kane's equations for a holonomic or nonholonomic system, all of which are minimal sets. In some situations, as described previously, forming and solving a nonminimal set of equations may be in order. In the event that constraint forces are of interest, one must either bring them into evidence according to Kane's instructions or employ equations in which the constraint equations are adjoined with Lagrange or undetermined multipliers. In the case of nonminimal equations, expressions for generalized constraint forces in F^c that do not involve λ are given as Eq. (32) of Ref. [42] and Eq. (96) of Ref. [6].

It is perhaps surprising that, despite the existence of methods for producing a minimal set of equations directly, in some cases multipliers are introduced into equations of motion and then removed. Significant effort has been devoted to eliminating the multipliers from dynamical equations and obtaining a minimal set of equations in the process. One way to accomplish this is to express in matrix form the equations containing the multipliers and then premultiply by an array containing an orthogonal complement to the constraint Jacobian matrix. In connection with Kane's equations with undetermined multipliers, the orthogonal complement is constructed via a zero-eigenvalue theorem by Kamman and Huston (Ref. [43]) as well as Wang and Huston (Ref. [79]), singular value decomposition by Singh and Likins (Ref. [68]), and successive multiplication of Householder transformations by Amirouche and Jia (Ref. [2]). When Kane's method has been used to form a minimal set of equations governing motions of a system that is not subject to some particular constraints, and subsequently those additional constraints must be taken into account, Wampler et al. (Ref. [78]) show how to reduce the equations to a new minimal set without ever introducing multipliers. For holonomic systems, Angeles and Lee (Ref. [3]) start with the Newton-Euler equations governing the motion of individual rigid bodies, adjoin scleronomic holonomic constraint equations with multipliers, and then use a natural orthogonal complement matrix to eliminate the multipliers and obtain a minimal set of equations. The analysis in Ref. [3] is extended in Ref. [64] so that it is applicable to nonholonomic systems; formation of the natural orthogonal complement is said to be computationally inexpensive compared to the zero-eigenvalue method and singular value decomposition. Generalized coordinate partitioning can be used to create an orthogonal complement that removes multipliers, if they are present, from Lagrange's equations for holonomic systems (Ref. [66], pp. 100, 125–132, and Ref. [18]).

1.2 Physical Meaning of the Multipliers

The Lagrange multipliers or undetermined multipliers contained in the column array λ are related to generalized constraint forces in the column array F^c through the constraint Jacobian α, $F^c = \alpha^{\mathrm{T}}\lambda$, as discussed in Sec. 1.1. It is just as important, if not more so, to understand the relationship of the multipliers to the actual constraint forces and torques acting in a mechanical system.

A set of generalized forces can be computed in a straightforward manner with knowledge of actual forces and torques; however, it is impractical to invert the process and determine actual forces and torques from generalized forces. In modeling a dynamical system, the applied or given forces are known functions of the generalized coordinates, the motion variables (or the time derivatives of the generalized coordinates), and the time t. In this sense each such force is considered to have a known direction and magnitude. These forces are typically regarded as equivalent to a set of single forces, each applied at a particular point, together with a set of couples, each with a torque applied to a particular rigid body. In the process of constructing the generalized forces needed for the methods of Lagrange or Kane, information about the direction, magnitude, and point or body of application of the forces and torques becomes lost; in principle, each generalized force is a sum of contributions from every force and torque in the aforementioned sets. With analytic expressions for the generalized forces in hand, one would be hard pressed to work backwards and determine the magnitude, direction, and point or body of application of every force and torque in the model. Likewise, one cannot readily recover such information about the actual constraint forces and constraint torques from analytic expressions for the elements of F^c. Difficulty in relating multipliers to constraint forces is recognized, for example, on p. 466 of Ref. [24] where the author instead recommends using free-body diagrams with all but the simplest of systems. Readers of Ref. [66] are alerted on pp. 133–134 that straightforward relationships between the multipliers and the actual reaction forces do not always exist. A similar problem is encountered when working with nonminimal equations and generalized constraint forces that are free of the multipliers; as suggested on p. 85 of Ref. [6], it is usually impossible to construct actual constraint forces if more than one contributes to F^c.

It is possible to make some progress in relating F^c to actual constraint forces by dealing with individual contributions to F^c from each constraint equation, rather than with F^c in its entirety. The column array F^c is a linear combination of the columns in α^{T}, where each column has an element of λ as its coefficient in the sum. Each constraint equation is associated with one column of α^{T} and one element of λ. The approach taken in Refs. [29] and [55] is to relate multipliers to constraint forces and torques by comparing the column array $\lambda_i(\partial\Phi_i/\partial\mathbf{q})^T$ to a column array Q_i of generalized constraint forces obtained by applying constraint forces at certain points and constraint torques to certain bodies. The drawback here is that in order to construct Q_i one must already know the constraint force and torque directions, and points and bodies of application. In Sec. 6.2.2 of Ref. [29] García de Jalón and Bayo explore the relationship, on a case by case basis, for some basic conditions such as constant distance between two points, a prismatic joint, a gear joint, and a constant angle between two lines. In a similar manner relationships between Lagrange multipliers and constraint forces associated with planar motion of a revolute joint, a revolute-revolute joint, and a prismatic joint are established in Sec. 9.3 of Ref. [55]. Nikravesh does not appear to consider motion constraints and nonholonomic constraint equations. Constraints dictated by a variety of joints, or kinematic pairs, are studied by Bauchau in Chapter 10 of Ref. [13]. Particular attention is paid to the revolute joint and the prismatic joint, for which a generalized constraint force array is written as a sum of column arrays with multipliers as coefficients. A relationship between a multiplier and the constraint forces and moments is inferred by examining the form of the corresponding column. Géradin and Cardona also devote a chapter of Ref. [31] to the study of configuration constraints imposed by joints used to connect members of a multibody system, and they include an examination of motion constraints associated with a rolling disk. The revolute and prismatic joints are examined in somewhat more detail than other joints, and in each case an expression for virtual work is inspected to relate multipliers to constraint forces. When it comes to motion constraints imposed on a vertical rolling disk, an interpretation of the multipliers is obtained by referring to the meaning of velocity level constraint equations rather than inspecting an expression for virtual work. Constraints involving prescribed motion are not addressed in Ref. [31] and consideration of the matter is left as an exercise for the reader in Ref. [13].

Kane's equations with undetermined multipliers are derived in Refs. [79] and [37]. In the course of the derivation, partial velocities, partial angular velocities, constraint forces, and constraint torques are all expressed in terms of a basis of unit vectors $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$, and $\hat{\mathbf{n}}_3$ fixed in an inertial reference frame; as a consequence, the undetermined multipliers become dot products of these unit vectors with the constraint forces applied to points whose velocity is prescribed, or with the constraint torques applied to bodies whose angular velocity is prescribed. In other words, the multipliers are measure numbers of constraint forces and torques for an inertial basis. In Ref. [79] Huston applies the method to a rolling disk, and three multipliers are measure numbers for an inertial basis of the constraint force applied to the disk at its point of contact with a plane. Although the clear definition of the multipliers in the general case is a welcome result, the additional expense of transformation is required if measure numbers for a non-inertial basis are desired. A related disadvantage is that working with all vectors in an inertial basis when obtaining equations of motion for a particular system is not necessarily convenient or efficient, and it is certainly not a requirement for applying Kane's method.

Dynamical equations of motion for a holonomic multibody system, analogous to those set forth in Ref. [79], are developed from Newton-Euler equations in Ref. [3]. The constraints of interest are those dictated by joints in a multibody system having the topography of a simple kinematic chain; they are thus represented by scleronomic equations and do not involve prescribed motion. Upon differentiation with respect to time, the constraint equations are expressed in terms of twist, a stack of the angular velocity of a rigid body in an inertial reference frame and the velocity of the body's mass center in an inertial reference frame. Specifically, the twist of a body is a 6×1 column array of measure numbers of the angular velocity and mass center's velocity for a basis of unit vectors fixed in that body. The counterpart to F^c, a column array of generalized constraint forces, is denoted by w^C and called the generalized nonworking constraint body wrench. The constraint forces acting on each body are regarded as equivalent to a single constraint force bound to the body's mass center, together with a couple that exerts a constraint torque on the body; w^C is simply a stack of body-basis measure numbers of the generic constraint torque and constraint force applied to each body. Thus in this instance there is a clear relationship between resultants of constraint forces and torques on the one hand, and the generalized constraint forces on the other, rather than the simple interpretation of the multipliers obtained in Refs. [79] and [37]. In general, each element of w^C is of course a linear combination of all of the Lagrange multipliers. Each generic constraint force and torque (body wrench) represented in w^C is, with some effort, related to the actual constraint force and torque (joint wrench) exerted by one body upon the next through the joint that connects them.

Lagrange's equations are invoked in Ref. [39] to study a three-link planar manipulator whose motion is completely prescribed by specifying the configuration of the third link; specifically, the two Cartesian coordinates of the tip and the orientation of the link are given as functions of time. The Lagrange multipliers are described straightforwardly, as they are in Ref. [79]. Two multipliers are inertial basis measure numbers of a constraint force applied to the tip, whereas the third multiplier is the inertial basis measure number of the torque of a constraint couple applied to the third link. These three multipliers are then related to the torques that would have to be supplied by motors acting at each of the three joints.

The foregoing review brings to light several reasons for dissatisfaction with existing approaches for relating multipliers to constraint forces and torques. First, they lack comprehensiveness to some degree; some cover configuration constraints but do not address motion constraints, and some do not consider prescribed motion specifically. A lack of uniformity also exists within Ref. [31]; a configuration constraint is treated by inspecting an expression for virtual work whereas a motion constraint is addressed by examining the form of a nonholonomic constraint equation. Second, the meaning of a multiplier is established in each of Refs. [13], [29], [31], and [55] on a case by case basis; these works do not offer a general procedure that can be used with a wide variety of constraints. Third, as has been pointed out, the comparison of column arrays performed in Refs. [29] and [55] depends for its success on prior knowledge of the direction and point of application of a constraint force, otherwise some guesswork or trial and error may be required. The avenue taken in Refs. [79] and [37] provides some measure of uniformity in that every multiplier is an inertial basis measure number of a constraint force or torque, but this result is unnecessarily restrictive as measure numbers for another basis may prove more convenient to the analyst and provide for a more efficient derivation of equations of motion. Although there exists a straightforward link between generalized constraint forces and constraint forces in Ref. [3], a clear correspondence between multipliers and constraint forces is absent, as is consideration of prescribed motion. An attempt in Ref. [64] to extend the approach to include motion constraints is limited to rolling. What is needed is a single method of relating a multiplier to constraint forces and torques; it should be uniform, general, and applicable to broad classes of motion constraints and configuration constraints, including prescribed motion.

1.3 A Critique of Current Methods

Upon closer examination, current concepts are found to suffer further shortcomings in addition to the deficiencies noted in Sec. 1.2.

Many of the problems are attributable to an unfortunate, pervasive tendency to rely on scalar analysis to an unnecessary extent instead of dealing with vectors whenever possible. It is important here to make a distinction between vectors and scalar representations of vectors. A vector is a basis-independent quantity, use of which frees the analyst from having to carry around scalar excess baggage. For example, the dot product of two vectors, $\mathbf{v} \cdot \mathbf{w}$, has meaning that is completely independent of whether one expresses the vectors in terms of a basis of unit vectors $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, and $\hat{\mathbf{a}}_3$ as $\mathbf{v} = v_1\hat{\mathbf{a}}_1 + v_2\hat{\mathbf{a}}_2 + v_3\hat{\mathbf{a}}_3$ and $\mathbf{w} = w_1\hat{\mathbf{a}}_1 + w_2\hat{\mathbf{a}}_2 + w_3\hat{\mathbf{a}}_3$, or in terms of a basis of unit vectors $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, and $\hat{\mathbf{b}}_3$ as $\mathbf{v} = V_1\hat{\mathbf{b}}_1 + V_2\hat{\mathbf{b}}_2 + V_3\hat{\mathbf{b}}_3$ and $\mathbf{w} = W_1\hat{\mathbf{b}}_1 + W_2\hat{\mathbf{b}}_2 + W_3\hat{\mathbf{b}}_3$. One may choose to represent $\mathbf{v} \cdot \mathbf{w}$ as $[\ v_1 \quad v_2 \quad v_3\ ][\ w_1 \quad w_2 \quad w_3\ ]^T$ or as $[\ V_1 \quad V_2 \quad V_3\ ][\ W_1 \quad W_2 \quad W_3\ ]^T$, but $\mathbf{v} \cdot \mathbf{w}$ has a meaning that does not depend on making either choice, or choosing any basis at all. In Ref. [55] a contrast is drawn between a "geometric" vector (p. 19) and an algebraic representation of the vector (p. 21). The latter consists of a column matrix whose scalar elements are measure numbers of the vector for a particular basis, whereas a vector has magnitude and direction that is independent of any basis. Reference [31] contains the recognition (p. 56) that a vector relationship is separate from its matrix analog.

With one possible exception García de Jalón and Bayo work in Ref. [29] with scalars rather than vectors in relating the multipliers to constraint forces. Their analysis is tied to scalar coordinates and, as a direct consequence, they must observe on pp. 229–230 that their approach to dealing with the multipliers is useful in connection with reference point coordinates and joint constraints, but not with natural coordinates and element constraints. The multipliers "directly provide the constraint forces" in the former case, but not in the latter. This conflict provides motivation for the use of vector quantities; it is advantageous precisely because one does not work with this or that set of scalar coordinates. Moreover, the distinction between element constraints and joint constraints is artificial and unnecessary.

The process of interpreting the meaning of the multipliers in individual situations by comparing two column arrays, as discussed in Sec. 1.2, is inherently a comparison of the scalar elements. The number of elements in each of the two arrays is equal to the number of generalized coordinates, which in turn is affected by whether a problem is planar or three-dimensional; the larger the arrays, the more work is required to demonstrate the relationship of the multiplier to the constraint forces and torques. It must be acknowledged on p. 227 of Ref. [29] that when working with scalars a three-dimensional case involves more mathematical complexity than a planar case. An expression involving vectors represents two- and three-dimensional situations with equal simplicity.

In dealing with a constant distance constraint in the planar case, $\lambda_i(\partial\Phi_i/\partial\mathbf{q})^T$ and Q_i are each 4×1 column arrays, whereas in the three-dimensional case they are 6×1. For the planar case alone, five equations are introduced in half a page in Ref. [29]. Upon examination of the right hand member of Eq. (6.95),

$$\lambda\{\ (x_i - x_j)\quad (y_i - y_j)\quad (z_i - z_j)\quad (x_j - x_i)\quad (y_j - y_i)\quad (z_j - z_i)\ \}$$

it is concluded that a constraint force is parallel to a bar; the same conclusion would be reached much more easily by inspecting a vector expression such as $\lambda L\hat{\mathbf{u}}$, where $L\hat{\mathbf{u}}$ is the position vector from one end of the bar to the other.

Study of a planar prismatic joint in Ref. [29] entails identifying each of two multipliers by comparing two 8×1 arrays; a total of four 12×1 arrays would be involved in three dimensions. One of the constraints is expressed as a requirement for two line segments to have equal slopes, which in turn drags several Cartesian coordinates into the analysis. This seems a convoluted way of dealing with the fundamental restriction: a point fixed in the slider cannot move in a direction perpendicular to the

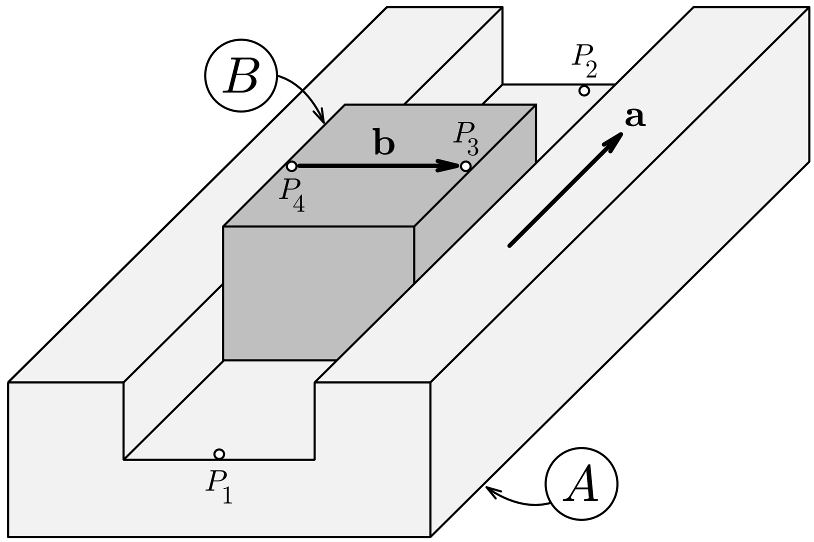


Figure 1: Planar Prismatic Joint

slider's track. Consider Figure 1 as an alternative to Fig. 6.11 of Ref. [29]. If A is a rigid body in which points P_1 and P_2 are fixed, in other words the body containing the track, and B is a rigid body in which points P_3 and P_4 are fixed, in other words the body that moves in the track, then the constraint can be expressed simply as $\mathbf{d} \cdot \mathbf{b} = 0$. The vector $\mathbf{b}$ is the position vector from P_4 to P_3, and $\mathbf{d}$ is the position vector from P_1 to a point of B that should remain collinear with P_1 and P_2. A second constraint, expressed in Eq. (6.79), involves a requirement for perpendicular orientation that could be written simply as $\mathbf{a} \cdot \mathbf{b} = 0$, where vectors $\mathbf{a}$ and $\mathbf{b}$ are fixed in A and B respectively. Instead, Eq. (6.79) is a scalar representation of a dot product and it involves Cartesian coordinates. The road to Eq. (6.81) leads through a scalar swamp; 8 equations (6.71)–(6.78) are compared to an 8×1 column in (6.80) to find out that the multiplier is given by $\lambda = M/(L_{12}L_{34})$.

Complications arising from the use of scalars can also be observed in Ref. [55]. The presence of direction cosine matrices in expressions for constraints requiring perpendicular vectors [Eqs. (7.3) and (7.4)] and parallel vectors [Eqs. (7.5) and (7.6)] shows that Nikravesh is working with basis-dependent scalars, not basis-independent vectors. Column matrices containing inertial (global) basis measure numbers lack prime superscripts, and those containing body (local) basis measure numbers are indicated with prime superscripts. As mentioned earlier, the use of vectors frees one from undue concern with coordinate systems and provides a contradiction to the statement found on p. 186: “In constraint equation formulation, it is necessary to express the components of all vectors in the same coordinate system, the most natural being the global coordinate system.” Likewise, an incorrect statement is made on p. 223: “It is possible to obtain a relationship between the constraint reaction forces and the constraint equations if (1) a proper vector of coordinates is defined and (2) the constraint forces are expressed with respect to the same coordinate system as the vector of coordinates.” Such relationships can be established without regard to coordinates.

The study of the parallel vector constraint in Sec. 7.1.2 of Ref. [55] furnishes a good example of the way in which problems associated with scalar analysis are eliminated if one works instead with vectors. In considering a restriction in which two vectors $\mathbf{v}$ and $\mathbf{w}$ must remain parallel, the requirement that the cross product of the two vanish, $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ is expressed with a scalar representation, and it is pointed out that one of the three associated scalar equations is linearly dependent on the other two. The local versions of the constraint equations are noticeably more complicated than the global versions; the advantage of expressing the cross product in terms of vectors is that it eliminates the need to distinguish between global and local bases, and the difference in complexity of the scalar expressions disappears. Furthermore, the use of vectors does away with the critical case identified by Nikravesh in which $\mathbf{v}$ and $\mathbf{w}$ become parallel to one of the global basis vectors; this is because the orientation of $\mathbf{v}$ and $\mathbf{w}$ relative to a third vector is of no importance in forming the vector product $\mathbf{v} \times \mathbf{w}$.

In a discussion of the universal joint on p. 190 of Ref. [55] it is proposed that, when body fixed coordinates are related to body fixed axes in certain ways, it is possible to simplify the matrix representation of the perpendicularity constraint imposed on the arms of the cross. By using the dot product of two vectors, $\mathbf{a} \cdot \mathbf{b} = 0$, the expression for the constraint is immediately in its simplest form; it is not possible to simplify any further this expression, which applies no matter how $\mathbf{a}$ is oriented in body A or how $\mathbf{b}$ is oriented in body B.

In Refs. [31] and [13] requirements for coincidence of two points and orthogonality of two unit vectors are expressed logically with vector expressions for dot products of vectors in mind, although Géradin and Cardona rely heavily on the matrix analog. For example, in analysis of a revolute joint each of Eqs. (7.55) contains a straightforward matrix analog to the dot product of two unit vectors; but subsequently there is a needless introduction of scalars in the form of a pair of direction cosine matrices in each of Eqs. (7.57).

Aside from the foregoing issues involving scalars and vectors, additional problems can be identified in current treatments of constrained motion.

Even though several constraint conditions are examined one at a time in Ref. [29] it is stated on pp. 233 and 238 that the Lagrange multiplier method is global, meaning that it requires calculation of all motor forces and reaction forces even when one might only be interested in a certain few. The virtual power method is suggested as a way of calculating constraint forces selectively; motor forces and reactions at the joints are treated with different approaches. A method that handles both types of constraints in a uniform, selective manner would represent a welcome improvement.

The analysis of a planar gear joint in Ref. [29] is bogged down in scalars, and unnecessarily burdened with transformations from inertial to wheel-fixed directions. A more serious deficiency, however, is the conclusion that a force is applied to the first wheel at point k (see Fig. 6.16). This is at odds with physical reasoning because there is nothing in contact with the wheel at k. A correct analysis of the problem should determine that the wheels exert a force on one another at their points of mutual

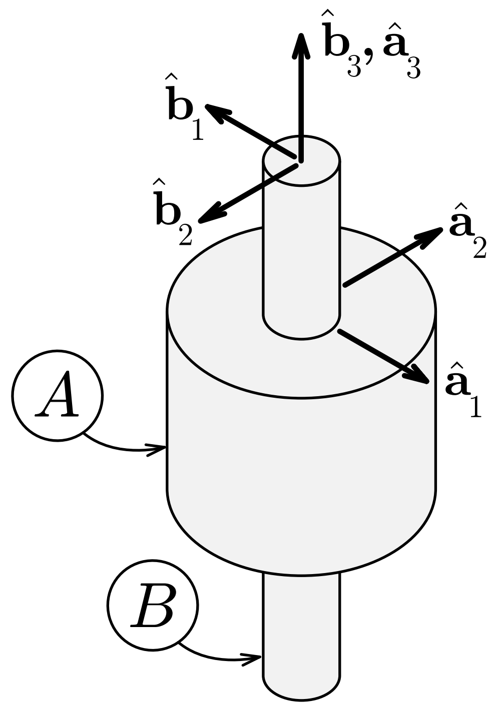


Figure 2: Cylindrical Joint

contact. The constraint equations (6.85) and (6.86) can be viewed as trigonometric identities relating the angles α_1, α_2, θ_1, and θ_2, but there are better ways of expressing the constraints. If one ignores details regarding interactions taking place at the gear teeth, the gear motion is fundamentally one of rolling, which is an absence of slipping. A concise statement of the constraint in vector form is ${}^N\mathbf{v}^{\tilde{A}} - {}^N\mathbf{v}^{\tilde{B}} = \mathbf{0}$, where $\tilde{A}$ and $\tilde{B}$ are points fixed in each wheel that are in contact during rolling motion, and where ${}^N\mathbf{v}^{\tilde{A}}$ and ${}^N\mathbf{v}^{\tilde{B}}$ are the velocities of the respective points in any reference frame N whatsoever.

Equations (7.7) of Ref. [55] express the constraint that two points P_1 and P_2, each belonging to separate bodies, must remain coincident as is the case in, for example, a spherical joint. The equations are unnecessarily complicated because, in addition to P_1 and P_2, they involve an origin on each body. The only two points of importance in this constraint are P_1 and P_2.

We return to the treatment in Ref. [55] of the constraint involving two parallel vectors to observe that it entails expenditure of wasted effort: three scalar equations are formed and one is discarded. Only two constraint equations are needed to express the restriction, therefore only two equations should be formed. Take for example the cylindrical joint shown here in Figure 2 and in Fig. 7.2 of Ref. [55]. Let A be the body that acts as the collar, and let B be the shaft on which A slides and turns. Introduce a dextral, mutually orthogonal set of unit vectors $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, and $\hat{\mathbf{a}}_3$ fixed in A, and a similar set $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, and $\hat{\mathbf{b}}_3$ fixed in B such that $\hat{\mathbf{a}}_3$ and $\hat{\mathbf{b}}_3$ are parallel to each other and to the axis of the shaft. Two equations expressing the requirement for parallelism can be written simply as $\hat{\mathbf{a}}_3 \cdot \hat{\mathbf{b}}_s = 0$ $(s = 1, 2)$. Two additional constraint equations are required for a complete description of a cylindrical joint; more effort is squandered with the second of Eqs. (7.13) by forming three more scalar constraint equations only to discard one. In this case it would be more efficient to construct just the two equations $\mathbf{d} \cdot \hat{\mathbf{b}}_s = 0$ $(s = 1, 2)$, where $\mathbf{d}$ is the position vector from point P_1 fixed in A on the axis of the shaft, to a similar point P_2 fixed in B. The inefficient formulation of the parallel vector constraint is employed for the revolute joint in Eq. (7.10), the prismatic joint in Eq. (7.15), and the screw joint in Eq. (7.16).

Planar motion of a revolute joint is examined on pp. 237–238 of Ref. [55]. Two Lagrange multipliers are related to constraint forces in the usual way by drawing free-body diagrams and carrying out the operations needed to evaluate $(\partial\Phi/\partial\mathbf{q})^T\lambda$. It is easily recognized that the multipliers are inertial basis measure numbers of a constraint force, but identifying the point of application requires interpretation of a moment equation. It would be advantageous to have at one's disposal a method that identifies the point of application immediately upon inspection of constraint equations, without the need to construct free-body diagrams or decipher a scalar moment equation involving Cartesian coordinates.

The process of Boolean identification of corresponding degrees of freedom in connection with the finite element approach is employed in Refs. [31] and [13]. In dealing with a revolute joint two constraints on relative orientation are handled with the Lagrange multiplier method while three constraints on relative translation are ignored due to Boolean identification. Conversely, in the case of a prismatic joint, the multiplier method is applied to two constraints on relative translation but three constraints on relative orientation are ignored. Even though computational efficiency is gained by disregarding certain constraints, a measure of inconsistency is introduced. In each case all five constraints ought to be treated in a uniform manner, and a constraint should be ignored when the associated constraint force is of no interest.

Derivations of Kane's equations with undetermined multipliers are performed in Refs. [79] and [37] by considering a specific type of constraint, namely prescribed motion, which is fundamentally a configuration constraint. Of course, a rheonomic holonomic constraint equation can be differentiated with respect to time and thereby cast in terms of prescribed velocity of a point or prescribed angular velocity of a rigid body. The first stated objective of Ref. [37] is to establish formally the validity of the relationship $F^c = \alpha^T \lambda$ for a general case in which the constraint equations can be expressed as linear combinations of time derivatives of the generalized coordinates; however, the proof involves the specific case of prescribed motion. For this reason, applicability of the resulting method to other configuration constraints such as those dictated by joints, and to motion constraints such as rolling, is open to question. Nevertheless, the equations are applied to a rolling disk in Ref. [79]. The proof would be strengthened considerably if it could be constructed with general constraint equations that are not limited to prescribed motion.

1.4 Using Vectors to Deal With Constraints

A dynamical equation that governs the motion of a mechanical system, whether subject to constraints or not, ultimately is expressed in terms of scalar quantities.

Nevertheless, use of basis-independent vectors and dyadics to derive equations of motion or establish relevant proofs is invariably more expeditious than the use of scalars, until a point is reached in the derivation when the introduction of scalars becomes unavoidable. The reader who employs vectors and dyadics to obtain equations of motion is no doubt familiar with the benefits of doing so whenever possible. Although this machinery is available for tackling general problems in dynamics, the established literature for dealing with constrained dynamical systems is presented almost exclusively in terms of scalar relationships. The disadvantages stemming from this state of affairs are twofold. First, one is denied any savings in labor that might be gained by the use of vectors and dyadics to deal with constraints. The second and more important issue is that the scalar point of view obscures useful vectorial information contained in the constraint equations with regard to constraint forces and torques.

Nonholonomic constraint equations, and holonomic constraint equations differentiated with respect to time, are expressed in the literature in terms of scalars, almost without exception; constraint equations at the velocity level are typically expressed in matrix form as $\alpha u + \beta = 0$, as discussed in Sec. 1.1. When one works with Lagrange's equations, a column array $\dot{q}$ containing time derivatives of generalized coordinates plays the part of u, which contains motion variables. Such scalar statements are devoid of the explicit appearance of vectors such as ${}^N\mathbf{v}^P$, the velocity of a particle P in an inertial reference frame N, and ${}^N\boldsymbol{\omega}^B$, the angular velocity of a rigid body B relative to N.

Analysis that elevates the discussion of constraints to the level of vectors has appeared only recently, and in very limited amount. Rosenthal and von Flotow provide a notable example in Ref. [77], where constraint equations at the velocity level are written in terms of dot products of ${}^N\mathbf{v}^P$ and ${}^N\boldsymbol{\omega}^B$ with other vectors. In addition, vectors such as $\mathbf{r}^{PQ}$, the position vector from a point P to a point Q, are brought into explicit evidence in holonomic constraint equations. Configuration constraints that involve restrictions on possible orientations are expressed as $\mathbf{a} \cdot \mathbf{b} = 0$, where vectors $\mathbf{a}$ and $\mathbf{b}$ are fixed in rigid bodies A and B respectively. It is suggested in Ref. [77] that the direction and point of application of a constraint force can be obtained by inspecting a constraint equation involving a dot product of ${}^N\mathbf{v}^P$ with another vector. Likewise, it is said that the direction of a constraint torque can be determined by inspecting a constraint equation in which the dot product of ${}^N\boldsymbol{\omega}^B$ with another vector appears, and that the angular velocity determines the body to which the constraint torque must be applied. However, a rigorous justification for these observations is lacking.

A second example is found in Ref. [27] where a constraint is expressed in terms of a dot product of a unit vector $\hat{\mathbf{a}}$ and the difference ${}^N\mathbf{v}^P - {}^N\mathbf{v}^{\overline{P}}$ of velocities of a pair of particles P and $\overline{P}$ that are momentarily or continuously in contact with each other. Equations of this form are said to describe the majority of constraints found in practice, including those represented by holonomic constraint equations (after differentiation) and nonholonomic constraint equations that are linear in the motion variables. The two particles obey the constraint, by hypothesis, if $\overline{P}$ exerts upon P a constraint force in the direction of $\hat{\mathbf{a}}$ and P exerts upon $\overline{P}$ a constraint force of equal magnitude and opposite direction. Again, a justification for the hypothesis is not explored. It is noted here that the number of particles whose velocities appear in dot products in an equation of constraint need not be limited to two; in principle, all of the particles in a system may be involved in a single constraint equation. For instance, a constraint of homogeneous strain can be imposed in a particle-and-spring model of a tether by requiring that equal distances be maintained between consecutive particles as discussed in Ref. [49]. This condition is described by a number of constraint equations, each involving three particles. One significant contribution of Ref. [27] is a procedure, to be used in connection with Kane's method, for choosing an additional motion variable such that a single constraint force measure number is brought into evidence only in the associated additional equation and not in any of the pre-existing or other additional equations.

Wang and Pao furnish a third example in Ref. [80]. A sum of dot products of inertial velocities with other vectors is used to express each equation of constraint for a system of particles, and the associated constraint forces acting on the particles are written in vector form. The work deals with constraint equations that are linear in velocity when written at the velocity level. Wang and Pao investigate the interrelationships of the variational equations of Jourdain, Gauss, Gibbs, and Appell to one another and to Newton's second law.

Two other works employ vectors in appreciable measure to deal with constrained motion. Angeles and Lee (Ref. [3]) adopt the Newton-Euler approach to forming equations of motion, and express holonomic constraint equations at the velocity level in terms of ${}^{N}\mathbf{v}^{B^\star}$ and ${}^{N}\boldsymbol{\omega}^{B}$ combined together in a stack and called the twist of a rigid body, where $B^\star$ is the mass center of B. Vector forms of constraint equations for revolute and prismatic joints are worked out in detail, but they contain cross products rather than dot products. The velocity constraint associated with a revolute joint involves the velocities of the mass centers of the bodies connected by the joint; as will be seen, it is simpler and more productive to write the constraint explicitly in terms of velocities of two points that must remain coincident. The work of Ref. [3] is extended in Ref. [64] to include the motion constraint of rolling, placed on an equal footing with configuration constraints in that both types are expressed at the velocity level in terms of twist; however, the development fails to include general cases of holonomic and nonholonomic constraint equations.

Other examples of the use of vectors can be found in Refs. [25] and [67] where some holonomic constraint equations, possibly rheonomic, are expressed as $\mathbf{a}\cdot\mathbf{b}-X(t)=0$ and the scalar $X(t)$ is a prescribed function of time. However, in these works the constraint equations are adjoined to the dynamical equations in the usual way by means of a scalar Jacobian matrix and scalar Lagrange multipliers, and velocity and angular velocity vectors never make an appearance. Reference [87] is concerned with the motion of a robot whose topography includes three kinematic loops. The torque applied by each of three motors is to be determined such that a point fixed in the payload of the robot moves as prescribed. A Jacobian that relates three joint speeds to three measure numbers of prescribed velocity is written in terms of position vectors, and the constraint equations for prescribed motion involve angular velocity vectors, but no attempt is made to relate the constraint torques (motor torques) to vectors appearing in the constraint equations.

Emphasis is placed in Refs. [6] and [23] on the benefits of working with scalar forms of constraint equations expressed at the acceleration level; when the constraints are written as $\alpha\dot{u} + \gamma = 0$ they have the column array of time derivatives of motion variables, $\dot{u}$, (or $\ddot{q}$ in the case of Lagrange's equations) in common with the dynamical equations of motion and this facilitates the solution for $\dot{u}$ and the multiplier array λ given in Sec. 1.1. The literature does not appear to contain any work in which acceleration appears explicitly in vector form in a constraint equation. One purpose of the present work is to show that a significant benefit attends the expression of a constraint equation at the acceleration level in terms of the vector ${}^N\mathbf{a}^P$, the inertial acceleration of a particle P. The advantage of doing so, as will be seen, is that the equation becomes a statement about the force necessary to enforce a constraint and this allows one to provide rigorous justification for the proposals regarding constraint forces made in Refs. [77] and [27].

1.5 Overview

The objective of this thesis is to formulate a comprehensive, consistent, and concise method for identifying a set of forces needed to constrain the behavior of a mechanical system. A comprehensive approach is one that is general enough to accommodate broad classes of motion constraints and configuration constraints, including prescribed motion. A consistent procedure is one that deals with all of the constraints in the classes of interest in a single, uniform way. In short, the goal is to develop a technique free of the deficiencies found in existing methods as described in Secs. 1.2 and 1.3.

A principal reason for the success of the method proposed here is that basis-independent vectors (as opposed to scalar representations and matrix analogs) are used in expressing constraint equations. This not only circumvents the pitfalls associated with scalars as noted in Sec. 1.3, it facilitates the use of Newton's second law in connection with constraint equations written in terms of acceleration vectors to permit a thorough validation of the relationships between constraint forces and constraint equations suggested in Refs. [77] and [27], and discussed briefly in Sec. 1.4.

The method can be summarized as follows. A constraint equation is, in general, written at the velocity level in terms of dot products such as ${}^N\mathbf{v}^P \cdot \mathbf{W}$ and ${}^N\boldsymbol{\omega}^B \cdot \boldsymbol{\tau}$, where ${}^N\mathbf{v}^P$ denotes the velocity of a particle P in an inertial reference frame N and ${}^N\boldsymbol{\omega}^B$ is the angular velocity of a rigid body B in N. The presence of ${}^N\mathbf{v}^P \cdot \mathbf{W}$ indicates that a constraint force parallel to $\mathbf{W}$ must be applied to P; the appearance of ${}^N\boldsymbol{\omega}^B \cdot \boldsymbol{\tau}$ means a constraint couple whose torque is parallel to $\boldsymbol{\tau}$ must be exerted on B. The parallel condition is met when the constraint force is given by $\lambda\mathbf{W}$ and the constraint torque by $\lambda\boldsymbol{\tau}$, where λ is an unknown scalar multiplier. The vectors $\mathbf{W}$ and $\boldsymbol{\tau}$ can be chosen, to some extent, at the convenience of the analyst.

Development of the method is presented in Chapter 2, beginning with a discussion of a solitary particle and progressing to consideration of the case in which subsets of particles make up rigid bodies. The results are thus uniformly applicable to all systems made up of particles and rigid bodies, subject to all configuration constraints described by independent holonomic constraint equations, including those involving prescribed positions, and all motion constraints that can be expressed with independent equations linear in velocity. One advantage of obtaining the constraint forces and torques in vector form is that they can be used together with any method for producing dynamical equations of motion that accommodates a force or torque having a known direction and an unknown magnitude. It is shown that constraint forces constructed with the proposed method are in fact noncontributing forces when one forms minimal sets of Kane's equations. Chapter 2 concludes with a discussion of the close relationship between the method of bringing constraint forces and torques into evidence by inspecting a constraint equation expressed in terms of dot products of vectors, and the procedure set forth in Ref. [44] for bringing measure numbers of noncontributing forces into evidence by the introduction of additional motion variables. As is the case with the latter procedure, the method proposed here can be applied selectively to bring into evidence only those constraint forces and torques that are of interest.

Comprehensiveness and consistency of the method are demonstrated in Chapters 3 and 4 by applying it to a wide range of constraints encountered in practice. A general approach for treating the configuration constraints dictated by joints is discussed first, followed by detailed examinations of a revolute joint, Hooke's joint, a spherical joint, and a prismatic joint. The motion constraint corresponding to the condition of rolling is studied next, and the third type of constraint to be considered is that of prescribed position. A second kind of motion constraint that involves a sharp-edged blade, and other configuration constraints including constant distance and prescribed position, are considered simultaneously in one example to show that the method can be applied to problems involving a variety of constraint types. Another example, presented in Ref. [44], is revisited to illustrate the relationship between the procedure given therein and the proposed technique; it is shown that the multipliers introduced by the latter method can be precisely the measure numbers brought into evidence by the former. Chapter 3 concludes with a demonstration that Newton's universal law of gravitation can follow from a treatment of two-body orbital motion in which Kepler's first and second laws are regarded as constraints to be satisfied.

Broadness in scope and utility are further illustrated in Chapter 4 by exercising the method with a problem of significant complexity, and carrying the analysis through to include a numerical solution of equations of motion. A robotic device, whose major parts are fastened together by seven revolute joints, is made to move a payload according to a prescribed schedule. The manipulator is modeled after an arm used today in construction of the International Space Station. It is kinematically redundant, meaning the number of revolute joints exceeds the number of coordinates, six, needed to describe the configuration of the payload with respect to the base of the manipulator. Relationships between the joint speeds and the velocity and angular velocity of the payload relative to the base are commonly referred to collectively as a resolved rate law; they are regarded here as constraint equations at the velocity level for expressing prescribed motion. Constraint forces and torques associated with the revolute joints are studied in a selective fashion; that is, interest is limited to the force involved at one joint and the torque present at a second joint. The chapter concludes with a presentation and discussion of results from a numerical simulation of a typical manipulator maneuver.

The method is even more comprehensive than it would appear after reading Chapters 3 and 4. Upon returning to the point in the development in Chapter 2 that deals with constraint equations at the acceleration level, the method can be extended to include constraint equations at the velocity level that are nonlinear in velocity; that is, nonlinear in the motion variables or the time derivatives of generalized coordinates. Although many motion constraints encountered in practice can be expressed with linear nonholonomic constraint equations, the literature contains several examples of servo-constraints or program constraints described by nonlinear nonholonomic constraint equations. In some cases linear equations are transformed into nonlinear ones.

The necessary modification to the method, as explained in Chapter 5, consists of inspecting a constraint equation written at the acceleration level instead of the velocity level, and identifying dot products such as ${}^N\mathbf{a}^Q \cdot \mathbf{W}$ and ${}^N\boldsymbol{\alpha}^B \cdot \boldsymbol{\tau}$, where ${}^N\mathbf{a}^Q$ is the acceleration of a point Q in a Newtonian reference frame N and where ${}^N\boldsymbol{\alpha}^B$ is the angular acceleration of a rigid body B in N. Presence of a dot product having the first form indicates a constraint force $\lambda\mathbf{W}$ is applied to Q, whereas the second form requires that a constraint torque $\lambda\boldsymbol{\tau}$ be exerted on B. Extension of the method is accompanied by the formulation of two new ways of deriving dynamical equations of motion. The first of these is used to produce dynamical equations that contain evidence of the constraint forces needed to satisfy nonlinear nonholonomic constraint equations, whereas constraint forces are not in evidence in the (minimal) equations of motion obtained in the second way. The novelty in each case rests in the use of partial accelerations rather than the partial velocities employed in Kane's method. The exposition begins in Chapter 5 with a system of particles and ends with consideration of the special case in which the system contains a rigid body. Equations of motion with and without evidence of constraint forces are obtained and solved numerically in four examples involving one or two particles subject to motion constraints expressed with equations that are inherently nonlinear in velocity. The question of whether or not an energy integral can be productively employed as a nonlinear nonholonomic constraint equation is entertained in the last section of the chapter; it is concluded that an integral of the motion cannot be used in this manner.

A recurrent mistake has been made and corrected in the literature many times over the past century; it has to do with the treatment of nonholonomic constraint equations within the framework of variational methods. The penultimate chapter of the thesis, Chapter 6, is devoted to the presentation of an additional argument to consider in this ongoing debate. Although it is known that correct dynamical equations of motion for a nonholonomic system cannot be obtained from a Lagrangean that has been augmented with a sum of the nonholonomic constraint equations weighted with multipliers, some publications suggest otherwise. Hagedorn points out in Ref. [34] that although such an approach is justified in the holonomic case, it is incorrect for nonholonomic constraint equations, even when they are linear. In an ensuing rebuttal, also given in Ref. [34], an example is proposed in support of augmentation and it purportedly demonstrates that an accepted method fails to produce correct equations of motion whereas augmentation leads to correct equations; Chapter 6 shows that in fact the opposite is true. The correct equations, previously discounted on the basis of a flawed application of the Newton-Euler method, are verified by using Kane's method together with the method for determining the directions of constraint forces. A correct application of the Newton-Euler method reproduces valid equations.

The thesis comes to a close in Chapter 7, where the reader finds statements of the major findings and important conclusions to be drawn from this work, as well as a discussion of possible avenues of continued investigation.

CHAPTER 2

RELATING CONSTRAINED MOTION TO FORCE

Force and acceleration are related by Newton's second law; therefore, it is possible to show that a holonomic or nonholonomic constraint equation expressed (after appropriate differentiation with respect to time) in terms of ${}^N\mathbf{a}^P$, the acceleration of a particle P in a Newtonian reference frame N, is a statement about the force that must be applied to P to impose the constraint. It is established in Sec. 2.1 that the presence in such a constraint equation of the dot product of ${}^N\mathbf{a}^P$ and a vector $\mathbf{W}$, ${}^N\mathbf{a}^P \cdot \mathbf{W}$, indicates a constraint force $\mathbf{C}$ must be applied to P such that it is parallel to $\mathbf{W}$; in other words, $\mathbf{C} = \lambda\mathbf{W}$. Hence, inspection of a constraint equation in which the dot product appears reveals that the point of application of the constraint force is P, and that the direction of $\mathbf{C}$ is parallel to the direction of $\mathbf{W}$. The magnitude of $\mathbf{C}$ can be determined once the scalar λ is known.

It very often happens that a constraint equation containing the dot product ${}^N\mathbf{a}^P \cdot \mathbf{W}$ is obtained by differentiating a constraint equation in which the dot product ${}^N\mathbf{v}^P \cdot \mathbf{W}$ appears. In this case it can be reasoned, as explained in Sec. 2.2, that knowledge of the direction and point of application of the constraint force $\mathbf{C} = \lambda\mathbf{W}$ is obtained by inspecting a constraint equation written at the velocity level rather than the acceleration level. When the velocity of only one particle appears in each dot product, the constraint equation is said to be linear in velocity or, what is the same, linear in the motion variables. All configuration constraints can be expressed with holonomic constraint equations that are linear at the velocity level. Motion constraints are always expressed at the outset with nonholonomic constraint equations at the velocity level, and linear equations can be used to describe important restrictions in this class, such as rolling (the absence of slipping). The material in Sec. 2.2 furnishes a rigorous justification for the proposals regarding constraint forces made in Refs. [77] and [27].

Constraint forces constructed with the present method are in fact noncontributing when one forms minimal sets of Kane's equations; the proof of this is supplied in Sec. 2.3. The constraint forces formed from nonholonomic constraint equations do not contribute to nonholonomic generalized active forces. Similarly, constraint forces fashioned from holonomic constraint equations don't contribute anything to holonomic generalized active forces, or to nonholonomic generalized active forces. Where holonomic systems are concerned, Lagrange's generalized forces are special cases of Kane's generalized active forces; consequently, the demonstration shows the present strategy leads to constraint forces that do not appear in a minimal set of Lagrange's equations for holonomic systems.

In practice one must be able to deal with a constrained system that contains one or more rigid bodies. The material in Sec. 2.4 makes it possible to do so. When a subset of particles makes up a rigid body B, use of the kinematic relation for their velocities shows that the appearance of the dot product ${}^{N}\boldsymbol{\omega}^{B} \cdot \boldsymbol{\tau}$ in a constraint equation written at the velocity level indicates a constraint couple must be applied to B such that the torque of the couple is parallel to $\boldsymbol{\tau}$. This proposal is explicit in Ref. [77] and hinted at in Ref. [27].

The present method of bringing constraint forces and torques into evidence by inspecting a constraint equation expressed in terms of dot products of vectors can be viewed as equivalent to the procedure set forth in Ref. [44] for bringing measure numbers of noncontributing forces into evidence by the introduction of additional motion variables. Introduction of an additional scalar motion variable is in effect the introduction of a constraint equation in terms of scalars at the velocity level. The present approach involves operating with vectors, whereas Kane's procedure takes place, for the most part, in terms of scalars. This relationship between the two processes is the subject of Sec. 2.5, which concludes the chapter. One advantage of a vectorial result is that it is not tied to a particular choice of motion variables, or even to the concept of a motion variable; therefore, the constraint forces and torques obtained by the present approach can be used together with any method of obtaining dynamical equations of motion that can treat a force or torque having a known direction and an unknown magnitude.

The chapter opens with a discussion of a solitary particle and proceeds by degrees to consideration of the case in which subsets of particles make up rigid bodies. The results, therefore, are applicable to all systems made up of particles and rigid bodies, subject to all configuration constraints described by independent holonomic constraint equations, and all motion constraints that can be expressed with independent equations linear in velocity.

2.1 Constraint Equations at the Acceleration Level

A constraint equation involving the acceleration of a particle in a Newtonian reference frame is, by virtue of Newton's second law, a statement about the resultant force applied to that particle. This idea is explored in Sec. 2.1.1, and extended in Sec. 2.1.2 to the case of two particles. Three important types of constraint involve a pair of particles, namely the requirements that they remain in contact with each other, that they are confined in a rigid body, and that two rigid bodies roll on each other without slipping. The concept is further generalized in Sec. 2.1.3 to include a system consisting of an arbitrary number of particles, and several simultaneous independent constraint equations are considered in Sec. 2.1.4.

2.1.1 A Single Particle

Newton's second law of motion for a particle P_1 states

$$\mathbf{R}_1 = m_1 \, {}^N\mathbf{a}^{P_1} \tag{1}$$

where $\mathbf{R}_1$ is the resultant of all contact and distance forces applied to P_1, m_1 is the mass of P_1, and ${}^N\mathbf{a}^{P_1}$ is the acceleration of P_1 in a Newtonian reference frame N. In view of Eq. (1), a limitation on the magnitude or direction of $\mathbf{R}_1$ implies and is implied by a limitation on the magnitude or direction of ${}^N\mathbf{a}^{P_1}$. For example, the direction of ${}^N\mathbf{a}^{P_1}$ can be prevented from being parallel to a vector $\mathbf{W}_1$; this constraint on the motion of P_1 can be expressed as

$${}^N\mathbf{a}^{P_1} \cdot \mathbf{W}_1 = 0 \tag{2}$$

and implies that $\mathbf{R}_1$ must also be constrained, as can be seen by dot-multiplying both sides of Eq. (1) with $\mathbf{W}_1$

$$\mathbf{R}_1 \cdot \mathbf{W}_1 = m_1 \, {}^N\mathbf{a}^{P_1} \cdot \mathbf{W}_1 \tag{3}$$

and then substituting from Eq. (2) into (3)

$$\frac{\mathbf{R}_1}{m_1} \cdot \mathbf{W}_1 = 0 \tag{4}$$

Thus, constraining the acceleration of P_1 can be accomplished by constraining the resultant of the forces applied to P_1. Equation (4) states that $\mathbf{R}_1$ must be perpendicular to $\mathbf{W}_1$; $\mathbf{R}_1$ can have no component parallel to $\mathbf{W}_1$.

Let $\mathbf{f}_1$ be the resultant of all contact and distance forces applied to P_1 when its motion is not restricted by the constraint expressed in Eq. (2). The acceleration of P_1 in N will not satisfy Eq. (2) when $\mathbf{f}_1$ has a component parallel to $\mathbf{W}_1$ (that is, when $\mathbf{f}_1 \cdot \mathbf{W}_1$ is not zero). However, the constraint is made possible through the application of another force $\mathbf{C}_1$, in which case the resultant $\mathbf{R}_1$ of the forces acting on P_1 is the sum of $\mathbf{f}_1$ and $\mathbf{C}_1$. Hence, Eq. (4) is rewritten

$$\frac{\mathbf{R}_1}{m_1} \cdot \mathbf{W}_1 = \frac{(\mathbf{f}_1 + \mathbf{C}_1)}{m_1} \cdot \mathbf{W}_1 = 0 \tag{5}$$

or

$$\mathbf{C}_1 \cdot \mathbf{W}_1 = -\mathbf{f}_1 \cdot \mathbf{W}_1 \tag{6}$$

Equation (6) can be satisfied as long as $\mathbf{C}_1$ has a component parallel to $\mathbf{W}_1$. Because a component of $\mathbf{C}_1$ perpendicular to $\mathbf{W}_1$ will not play a part in Eq. (6) it is sufficient for $\mathbf{C}_1$ itself to be parallel to $\mathbf{W}_1$, a condition that can be expressed as

$$\mathbf{C}_1 = \lambda \mathbf{W}_1 \tag{7}$$

where λ, a scalar, is determined by substituting from Eq. (7) into (6):

$$\lambda = -\frac{\mathbf{f}_1 \cdot \mathbf{W}_1}{{\mathbf{W}_1}^2} \tag{8}$$

A constraint on the motion of P_1 can be expressed in a form more general than that of Eq. (2),

$${}^{N}\mathbf{a}^{P_1} \cdot \mathbf{W}_1 + Z = 0 \tag{9}$$

where Z is a scalar. Under these circumstances Eq. (5) gives way to

$$\frac{\mathbf{R}_1}{m_1} \cdot \mathbf{W}_1 + Z = \frac{(\mathbf{f}_1 + \mathbf{C}_1)}{m_1} \cdot \mathbf{W}_1 + Z = 0 \tag{10}$$

Only a component of $\mathbf{C}_1$ that is parallel to $\mathbf{W}_1$ will play a part in Eq. (10), therefore Eq. (7) remains applicable and Eq. (8) is replaced by

$$\lambda = -\frac{Z + (\mathbf{f}_1 \cdot \mathbf{W}_1)/m_1}{{\mathbf{W}_1}^2/m_1} \tag{11}$$

It has been shown that a constraint expressed in the form of Eq. (9) can be achieved through the application of a force given by Eq. (7). Hence, one can inspect Eq. (9), then proceed immediately to write Eq. (7), and say that in general $\mathbf{C}_1$ must be applied to P_1 in order for the constraint to be satisfied.

2.1.2 Two Particles

The motion of every member of a system of particles is governed by Newton's second law; therefore, a counterpart to Eq. (1) can be written for a particle P_2

$$\mathbf{R}_2 = m_2 \, {}^N\mathbf{a}^{P_2} \tag{12}$$

Through the application of forces, the accelerations in N of P_1 and P_2 can be restricted. For example, a constraint involving both particles can be expressed as

$${}^N\mathbf{a}^{P_1} \cdot \mathbf{W}_1 + {}^N\mathbf{a}^{P_2} \cdot \mathbf{W}_2 + Z = 0 \tag{13}$$

By virtue of Newton's second law the resultants of the forces applied to P_1 and P_2 are then also restricted, as can be seen by substituting from Eqs. (1) and (12) into (13).

$$\frac{\mathbf{R}_1}{m_1} \cdot \mathbf{W}_1 + \frac{\mathbf{R}_2}{m_2} \cdot \mathbf{W}_2 + Z = 0 \tag{14}$$

Let $\mathbf{f}_1$ and $\mathbf{f}_2$ be the respective resultants of the forces applied to P_1 and P_2 when motions of the two particles are not restricted as described in Eq. (13). A constraint can be imposed by applying to P_1 a force $\mathbf{C}_1$, and to P_2 a force $\mathbf{C}_2$. The resultant of the forces acting on P_1 is then given by $\mathbf{R}_1 = \mathbf{f}_1 + \mathbf{C}_1$ and the resultant of the forces acting on P_2 is $\mathbf{R}_2 = \mathbf{f}_2 + \mathbf{C}_2$. Thus, Eq. (14) can be expressed as

$$\frac{(\mathbf{f}_1 + \mathbf{C}_1)}{m_1} \cdot \mathbf{W}_1 + \frac{(\mathbf{f}_2 + \mathbf{C}_2)}{m_2} \cdot \mathbf{W}_2 + Z = 0 \tag{15}$$

Any component of $\mathbf{C}_1$ perpendicular to $\mathbf{W}_1$ will not play a part in Eq. (15), nor will any component of $\mathbf{C}_2$ perpendicular to $\mathbf{W}_2$; consequently, all that is required of $\mathbf{C}_1$ is that it be parallel to $\mathbf{W}_1$, and $\mathbf{C}_2$ need only be parallel to $\mathbf{W}_2$.

$$\mathbf{C}_1 = \lambda_1 \mathbf{W}_1, \qquad \mathbf{C}_2 = \lambda_2 \mathbf{W}_2 \tag{16}$$

Reference [27] is concerned with constraint equations involving pairs of particles,

$$({}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1}) \cdot \mathbf{W} + Y = 0 \tag{17}$$

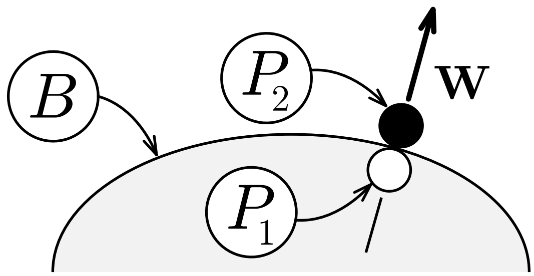


Figure 3: Two Particles in Contact

Equations written in this form are said to represent the majority of constraints encountered in practice. In fact, with $Y = 0$, they embrace three important types of constraint. The first two types place limits on the configuration of P_1 and P_2; they can be described with equations that involve position vectors from a point fixed in N, and that can be differentiated to be brought into this form. The two situations of interest include two particles that must remain in contact with each other, and two particles confined in a rigid body. Two particles that are in contact when two rigid bodies roll on each other without slipping constitute the third type of constraint, which is a motion constraint expressible as Eq. (17) at the outset. If $\mathbf{W}$ is fixed in, say, a reference frame B, differentiation of Eq. (17) with respect to time in N yields

$$({}^N\mathbf{a}^{P_2} - {}^N\mathbf{a}^{P_1}) \cdot \mathbf{W} + ({}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1}) \cdot ({}^N\boldsymbol{\omega}^B \times \mathbf{W}) = 0 \tag{18}$$

and this is recognized to have the form of Eq. (13) when $\mathbf{W}$ plays the dual role of $\mathbf{W}_2$ and $-\mathbf{W}_1$, and Z is defined as $Z \triangleq ({}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1}) \cdot ({}^N\boldsymbol{\omega}^B \times \mathbf{W})$. The law of action and reaction dictates that in each of the three types of constraint only one multiplier per constraint equation is required, not two as indicated in Eq. (16). In connection with the first type of constraint, illustrated in Figure 3, let particle P_2 be in contact with a rigid body B having a smooth surface, regard P_1 as the particle of B in contact with P_2, and take $\mathbf{W}$ to be normal to the surface of B at P_1. The law of action and reaction asserts that the force exerted by P_2 on P_1 and the force exerted by P_1 on P_2 have equal magnitudes, opposite directions, and coincident lines of action. We assume Eq. (18) is satisfied via physical contact of P_2 and P_1, and hence application of the law in the direction of $\mathbf{W}$ requires $\mathbf{C}_2 = -\mathbf{C}_1$, or $\lambda_2\mathbf{W} = -\lambda_1(-\mathbf{W}) = \lambda_1\mathbf{W}$, thus $\lambda_2 = \lambda_1 = \lambda$. The second type of constraint confines the behavior of two particles P_1 and P_2 that belong to a rigid body B. Their velocities in N are related by ${}^N\mathbf{v}^{P_2} = {}^N\mathbf{v}^{P_1} + {}^N\boldsymbol{\omega}^B \times \mathbf{W}$ where $\mathbf{W}$ is the position vector from P_1 to P_2; it is clear that Eq. (17) results from forming the dot product of both sides of this equation with $\mathbf{W}$ and setting $Y = 0$. The law of action and reaction dictates that the resultant $\mathbf{C}_2$ of all contact and distance forces exerted by P_1 on P_2 is parallel to $\mathbf{W}$, with a magnitude equal to and a direction opposite of the resultant $\mathbf{C}_1$ of all contact and distance forces exerted by P_2 on P_1. Again the consequence is that the multipliers λ_2 and λ_1 are identical. In the third type of constraint particles P_1 and P_2 are in rolling contact and belong to rigid bodies B and B' respectively; the condition of rolling can be expressed as ${}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1} = \mathbf{0}$, giving rise to three relationships wherein one of three mutually perpendicular vectors plays the part of $\mathbf{W}$. When $\mathbf{W}$ is parallel to the plane of contact of B and B', the constraint equation (17) is nonholonomic. The law of action and reaction leads once again to $\lambda_2 = \lambda_1 = \lambda$ for each constraint equation; of course, the multipliers associated with each of the three equations are different from one another.

It is the differentiation of Eq. (17) to bring ${}^N\mathbf{a}^{P_1}$ and ${}^N\mathbf{a}^{P_2}$ into explicit evidence, together with an appeal to Newton's second and third laws, that now provides a firm justification for the supposition in Ref. [27] that a constraint equation at the velocity level can be imposed by application of constraint forces applied to P_1 and P_2, given by

$$\mathbf{C}_2 = \lambda\mathbf{W}, \qquad \mathbf{C}_1 = -\lambda\mathbf{W} \tag{19}$$

for the three types of constraints just examined.

When only one multiplier is needed to characterize $\mathbf{C}_1$ and $\mathbf{C}_2$, Eq. (15) gives way to

$$\frac{(\mathbf{f}_2 + \lambda \mathbf{W}_2)}{m_2} \cdot \mathbf{W}_2 + \frac{(\mathbf{f}_1 + \lambda \mathbf{W}_1)}{m_1} \cdot \mathbf{W}_1 + Z = 0 \tag{20}$$

and λ is given by

$$\lambda = -\frac{Z + (\mathbf{f}_1 \cdot \mathbf{W}_1)/m_1 + (\mathbf{f}_2 \cdot \mathbf{W}_2)/m_2}{\mathbf{W}_1{}^2/m_1 + \mathbf{W}_2{}^2/m_2} \tag{21}$$

2.1.3 A Multiparticle System

The foregoing discussion involving two particles is easily extended to a system containing an arbitrary number of particles. In a system S made up of particles $P_1, \ldots, P_\nu$ the motion of every particle is governed by Newton's second law and

$$\mathbf{R}_i = m_i \, {}^N\mathbf{a}^{P_i} \qquad (i = 1, \ldots, \nu) \tag{22}$$

Through the application of forces the accelerations in N of $P_1, \ldots, P_\nu$ can be restricted in some manner; for example,

$$\sum_{i=1}^{\nu} {}^N\mathbf{a}^{P_i} \cdot \mathbf{W}_i + Z = 0 \tag{23}$$

where Z is a scalar. By virtue of Newton's second law the resultants of the forces applied to each particle are then also restricted as can be seen by substituting from Eqs. (22) into (23).

$$\sum_{i=1}^{\nu} \frac{\mathbf{R}_i}{m_i} \cdot \mathbf{W}_i + Z = 0 \tag{24}$$

Let $\mathbf{f}_1, \ldots, \mathbf{f}_\nu$ be the respective resultants of forces applied to $P_1, \ldots, P_\nu$ when their accelerations in N are not constrained in the manner set forth in Eq. (23). The system can be compelled to move in accordance with the constraint through the application of forces $\mathbf{C}_1, \ldots, \mathbf{C}_\nu$ to $P_1, \ldots, P_\nu$ respectively. The resultant of the forces acting on P_i is then given by $\mathbf{R}_i = \mathbf{f}_i + \mathbf{C}_i$ $(i = 1, \ldots, \nu)$. Thus, Eq. (24) can be expressed as

$$\sum_{i=1}^{\nu} \frac{(\mathbf{f}_i + \mathbf{C}_i)}{m_i} \cdot \mathbf{W}_i + Z = 0 \tag{25}$$

Any component of $\mathbf{C}_i$ perpendicular to $\mathbf{W}_i$ will not play a part in Eq. (25); consequently, all that is required of $\mathbf{C}_i$ is that it be parallel to $\mathbf{W}_i$. If we restrict the discussion to constraint forces $\mathbf{C}_1, \ldots, \mathbf{C}_\nu$ that have a single scalar in common,

$$\mathbf{C}_i = \lambda \mathbf{W}_i \qquad (i = 1, \ldots, \nu) \tag{26}$$

then λ is obtained by substituting from Eqs. (26) into (25)

$$\lambda = -\frac{Z + \sum_{i=1}^{\nu} \left(\mathbf{f}_i \cdot \mathbf{W}_i\right)/m_i}{\sum_{i=1}^{\nu} {\mathbf{W}_i}^2/m_i} \tag{27}$$

When $\mathbf{R}_i = \mathbf{f}_i + \mathbf{C}_i$ $(i = 1, \ldots, \nu)$, $\mathbf{C}_i$ is given by Eqs. (26), and λ is equal to the right hand member of Eq. (27), the motion of $P_1, \ldots, P_\nu$ is constrained as described in Eq. (23). For this reason one can inspect Eq. (23), immediately write Eqs. (26), and say that in general $\mathbf{C}_1, \ldots, \mathbf{C}_\nu$ must be applied in order for Eq. (23) to be obeyed.

2.1.4 Several Constraints

The material in Sec. 2.1.3 can be used to treat a system made up of an arbitrary number of particles, subject to one constraint. The scope of what has been presented is readily broadened to include more than one constraint by giving the scalar Z a subscript, and by giving a second subscript to the vectors $\mathbf{W}_i$ and $\mathbf{C}_i$. When there are m independent constraint equations expressible in the form of Eq. (23) they are written as

$$\sum_{i=1}^{\nu} {}^{N}\mathbf{a}^{P_i} \cdot \mathbf{W}_{is} + Z_s = 0 \qquad (s = 1, \ldots, m) \tag{28}$$

Substitution from Eqs. (22) into (28) leads to

$$\sum_{i=1}^{\nu} \frac{\mathbf{R}_i}{m_i} \cdot \mathbf{W}_{is} + Z_s = 0 \qquad (s = 1, \ldots, m) \tag{29}$$

Let $\mathbf{f}_i$ be the resultant of the contact and distance forces applied to P_i when the accelerations in N of $P_1, \ldots, P_\nu$ are not restricted as described in Eqs. (28). It is possible to satisfy these constraints through the application of forces $\mathbf{C}_{i1}, \ldots, \mathbf{C}_{im}$ to

P_i, in which case the resultant of all forces acting on P_i is given by $\mathbf{R}_i = \mathbf{f}_i + \sum_{s=1}^{m} \mathbf{C}_{is}$ $(i = 1, \ldots, \nu)$. Thus, Eqs. (29) can be expressed as

$$\sum_{i=1}^{\nu} \frac{\left(\mathbf{f}_i + \sum_{j=1}^{m} \mathbf{C}_{ij}\right)}{m_i} \cdot \mathbf{W}_{is} + Z_s = 0 \qquad (s = 1, \ldots, m) \tag{30}$$

Any component of $\mathbf{C}_{is}$ perpendicular to $\mathbf{W}_{is}$ will not play a part in the sth of Eqs. (30); consequently, all that is required of $\mathbf{C}_{is}$ is that it be parallel to $\mathbf{W}_{is}$. As in Sec. 2.1.3, we confine our attention to forces $\mathbf{C}_{is}$ $(i = 1, \ldots, \nu)$ that have a single scalar λ_s in common $(s = 1, \ldots, m)$ and write

$$\mathbf{C}_{is} = \lambda_s \mathbf{W}_{is} \qquad (i = 1, \ldots, \nu;\; s = 1, \ldots, m) \tag{31}$$

Substitution from Eqs. (31) into (30) leads to the m relationships

$$\sum_{i=1}^{\nu} \frac{\left(\mathbf{f}_i + \sum_{j=1}^{m} \lambda_j \mathbf{W}_{ij}\right)}{m_i} \cdot \mathbf{W}_{is} + Z_s = 0 \qquad (s = 1, \ldots, m) \tag{32}$$

that can, in principle, be solved for the multipliers. If λ is defined as an $m \times 1$ column matrix whose elements are $\lambda_1, \ldots, \lambda_m$, the solution is given by

$$\lambda = Q^{-1} R \tag{33}$$

where Q is an $m \times m$ matrix whose elements are defined as

$$Q_{sj} \triangleq \sum_{i=1}^{\nu} \frac{\mathbf{W}_{is} \cdot \mathbf{W}_{ij}}{m_i} \qquad (s, j = 1, \ldots, m) \tag{34}$$

and R is an $m \times 1$ column matrix whose elements are

$$R_s \triangleq -\left(Z_s + \sum_{i=1}^{\nu} \frac{\mathbf{f}_i \cdot \mathbf{W}_{is}}{m_i}\right) \qquad (s = 1, \ldots, m) \tag{35}$$

It is usually preferable to regard a system as composed of a finite number of rigid bodies and particles, rather than an arbitrary number of solitary particles. Subsequent developments will put us into position to deal with rigid bodies and determine the multipliers without forming Q and R; therefore, we put aside further discussion of the formation of those matrices and the existence of the inverse of Q.

In any case one can inspect Eqs. (28), immediately write Eqs. (31), and say that in general $\mathbf{C}_{is}$ $(s = 1, \ldots, m)$ must be applied to P_i $(i = 1, \ldots, \nu)$ in order to satisfy the constraints expressed in Eqs. (28).

2.2 Configuration and Motion Constraints

The preceding material establishes that when a constraint equation contains the dot product ${}^{N}\mathbf{a}^{P} \cdot \mathbf{W}$, one may inspect the constraint equation and conclude that a constraint force parallel to $\mathbf{W}$ must be applied to P if the constraint is to be obeyed. A constraint equation at the acceleration level very often arises from a constraint equation at the velocity level in which the velocity of only one particle appears in any dot product; it is shown in what follows that in this case one may inspect the latter equation instead of the former to identify the direction of a constraint force and the particle to which it must be applied. Section 2.2.1 undertakes an exploration of the connection between a configuration constraint, described at the position level by a holonomic constraint equation, and the resulting constraint equations at the velocity and acceleration levels. Section 2.2.2 contains an examination of the connection between a motion constraint expressed with a nonholonomic equation that is linear in velocity, and the concomitant constraint equation at the acceleration level. Because holonomic constraint equations differentiated with respect to time, as well as linear nonholonomic constraint equations, can be expressed with dot products of the form ${}^{N}\mathbf{v}^{P} \cdot \mathbf{W}$, forces needed to impose both types of constraints can be treated in the same way.

2.2.1 Holonomic Constraint Equations

The configuration of a system S of particles $P_1, \ldots, P_\nu$ in a Newtonian reference frame N is completely specified when all position vectors $\mathbf{p}_i$ from a point O fixed in N to P_i are known $(i = 1, \ldots, \nu)$. Holonomic constraint equations can be expressed (Ref. [44],

p. 35) in terms of 3ν scalar coordinates and the time t,

$$f_s(q_1, \ldots, q_{3\nu}, t) = 0 \qquad (s = 1, \ldots, M) \tag{36}$$

The dependency of M of the coordinates on the remaining $3\nu - M$ coordinates is implicit in these M relationships. However, the equations expressing restrictions on the configuration of S in N can also be written in terms of the ν position vectors $\mathbf{p}_i$ and t.

$$g_s(\mathbf{p}_1, \ldots, \mathbf{p}_\nu, t) = f_s(q_1, \ldots, q_{3\nu}, t) = 0 \qquad (s = 1, \ldots, M) \tag{37}$$

The position vectors are vector functions of $q_1, \ldots, q_{3\nu}$ and t in N. The explicit appearance of t in any of the vector functions $\mathbf{p}_1, \ldots, \mathbf{p}_\nu$ or in g_s renders a holonomic constraint equation rheonomic; otherwise, it is scleronomic.

The holonomic constraint equations (37) can be brought to the velocity level by differentiation with respect to t; although the scalar functions g_s are not differentiated with respect to t in any particular reference frame, differentiation of the vectors $\mathbf{p}_1, \ldots, \mathbf{p}_\nu$ must be carried out with respect to a common reference frame and it is convenient to choose N for this purpose. The derivatives are given by

$$\frac{dg_s}{dt} = \sum_{i=1}^{\nu} \frac{\partial g_s}{\partial \mathbf{p}_i} \cdot \frac{{}^N d}{dt}\mathbf{p}_i \; + \; \frac{\partial g_s}{\partial t} = 0 \qquad (s = 1, \ldots, M) \tag{38}$$

Differentiation of a scalar function G with respect to a vector $\mathbf{v}$ is the subject of Sec. 2.9 in Ref. [46]; $\partial G/\partial \mathbf{v}$ is sometimes denoted by $\boldsymbol{\nabla}_{\mathbf{v}}\, G$. It is important to note that $\partial g_s/\partial \mathbf{p}_i$ is a vector function of $q_1, \ldots, q_{3\nu}$ and t in N, and that $\partial g_s/\partial t$ is a scalar function of the same variables. It is also crucial to recognize ${}^N d\mathbf{p}_i/dt$ is the velocity ${}^N\mathbf{v}^{P_i}$ of P_i in N. Equations (38) are rewritten as

$$\sum_{i=1}^{\nu} {}^N\mathbf{v}^{P_i} \cdot \mathbf{W}_{is} + Y_s = 0 \qquad (s = 1, \ldots, M) \tag{39}$$

where $\mathbf{W}_{is}$ and Y_s, defined as

$$\mathbf{W}_{is} \triangleq \frac{\partial g_s}{\partial \mathbf{p}_i}, \qquad Y_s \triangleq \frac{\partial g_s}{\partial t} \qquad (s = 1, \ldots, M;\; i = 1, \ldots, \nu) \tag{40}$$

are, respectively, vector functions of $q_1, \ldots, q_{3\nu}$ and t in N, and scalar functions of the same variables. As $\mathbf{W}_{is}$ is not a vector function of any time derivative of a generalized coordinate, it is prohibited from being the velocity of any particle in any reference frame. Hence, Eqs. (39) are said to be linear in velocity. The configuration constraints expressed at the velocity level in Eqs. (39) involve, in general, the velocities in N of all particles belonging to S; the equations given in Ref. [27] include the velocities of only two particles and are thus special cases (albeit important ones) of the general form.

The rigid nature of a body B is a consequence of restrictions placed on the positions of particles belonging to B; the distance between any two particles P_1 and P_2 is required to remain constant. If the distance is denoted by L, the holonomic constraint equation that confines the positions of each pair of particles can be written in the form of Eqs. (37) as $g = (\mathbf{p}_2 - \mathbf{p}_1) \cdot (\mathbf{p}_2 - \mathbf{p}_1) - L^2 = 0$. Differentiation of this equation can be carried out according to Eqs. (38) by forming the quantities $\partial g/\partial \mathbf{p}_2 = 2\underline{\mathbf{U}} \cdot (\mathbf{p}_2 - \mathbf{p}_1)$, $\partial g/\partial \mathbf{p}_1 = -2\underline{\mathbf{U}} \cdot (\mathbf{p}_2 - \mathbf{p}_1)$, and $\partial g/\partial t = 0$, where $\underline{\mathbf{U}}$ is the unit dyadic; hence, the constraint at the velocity level has the form of Eqs. (39) and is given by $2({}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1}) \cdot (\mathbf{p}_2 - \mathbf{p}_1) = 0$, in agreement with the discussion in Sec. 2.1.2.

Differentiation of Eqs. (39) with respect to t in N brings the holonomic constraint equations to the acceleration level

$$\sum_{i=1}^{\nu} {}^N\mathbf{a}^{P_i} \cdot \mathbf{W}_{is} + Z_s = 0 \qquad (s = 1, \ldots, M) \tag{41}$$

where $Z_s \triangleq \dot{Y}_s + \sum_{i=1}^{\nu} {}^N\mathbf{v}^{P_i} \cdot {}^N d\mathbf{W}_{is}/dt$ are scalar functions of $q_1, \ldots, q_{3\nu}, \dot{q}_1, \ldots, \dot{q}_{3\nu}$ and t. Equations (41) are identical in form to Eqs. (28) and are seen to be derived from Eqs. (39); because the vectors $\mathbf{W}_{is}$ appear in the constraint equations expressed at the velocity level and the acceleration level, one may inspect relationships having the form of Eqs. (39), rather than Eqs. (41), and determine that constraint forces

$\mathbf{C}_{is} = \lambda_s \mathbf{W}_{is}$ must be applied to P_i to effect a configuration constraint. The vectors $\mathbf{W}_{is}$ appearing in Eqs. (28) are all required to posses the same units for a given value of s but can otherwise be any vectors whatsoever, and Z_s can be any scalar having the same units as the dot products ${}^N\mathbf{a}^{P_i} \cdot \mathbf{W}_{is}$. However, it has been shown that $\mathbf{W}_{is}$, Y_s, and Z_s have specific functional characters when Eqs. (39) and (41) result from configuration constraints.

2.2.2 Linear Nonholonomic Constraint Equations

When M independent configuration constraints are imposed on a system S of particles $P_1, \ldots, P_\nu$, the configuration of S in N is uniquely described by $n \triangleq 3\nu - M$ generalized coordinates $q_1, \ldots, q_n$ for S in N. The system S may also be subject to motion constraints, and the equations describing them are often linear in the motion variables $u_1, \ldots, u_n$ (Ref. [44], p. 40) which are themselves linear combinations of $\dot{q}_1, \ldots, \dot{q}_n$. In general m such equations that are independent of one another can be written in vector form as

$$\sum_{i=1}^{\nu} {}^N\mathbf{v}^{P_i} \cdot \mathbf{W}_{is} + Y_s = 0 \qquad (s = 1, \ldots, m) \tag{42}$$

where $\mathbf{W}_{is}$ are vector functions of $q_1, \ldots, q_n$ and t in N, and Y_s are scalar functions of $q_1, \ldots, q_n$ and t. Although Eqs. (39) bear a resemblance to Eqs. (42), the latter cannot be obtained by differentiating relationships that involve $\mathbf{p}_i$, such as Eqs. (37), and hence Eqs. (42) are referred to as nonholonomic. When S is subject to m independent motion constraints it is referred to as a nonholonomic system possessing $p \triangleq n - m$ degrees of freedom in N (Ref. [44], p. 43).

Equations (42) can be differentiated with respect to t, with each vector differentiated with respect to t in frame N, to bring the nonholonomic constraint equations to the acceleration level

$$\sum_{i=1}^{\nu} {}^N\mathbf{a}^{P_i} \cdot \mathbf{W}_{is} + Z_s = 0 \qquad (s = 1, \ldots, m) \tag{43}$$

where $Z_s \triangleq \dot{Y}_s + \sum_{i=1}^{\nu} {}^N\mathbf{v}^{P_i} \cdot {}^N d\mathbf{W}_{is}/dt$ are scalar functions of $u_1, \ldots, u_n$ (or $\dot{q}_1, \ldots, \dot{q}_n$), $q_1, \ldots, q_n$, and t. The comments regarding constraint forces made in connection with configuration constraints apply to motion constraints as well. In particular, the vectors $\mathbf{W}_{is}$ appear in the constraint equations expressed at the velocity level and the acceleration level; therefore one may inspect relationships having the form of Eqs. (42) rather than Eqs. (43), and determine that constraint forces $\mathbf{C}_{is}$ with directions parallel $\mathbf{W}_{is}$ $(s = 1, \ldots, m)$ must be applied to P_i $(i = 1, \ldots, \nu)$ to effect a motion constraint.

Because Eqs. (42) have the same form as Eqs. (39), the forces needed to bring about configuration constraints and motion constraints can be treated in a unified way when both types of constraints are expressed in terms of dot products of the velocities in N of the particles belonging to S. An important advantage of writing Eqs. (42) and Eqs. (39) in vector form, rather than the usual scalar form, is that they are independent of the choice of motion variables; in fact, the concept of motion variables can be put aside. Consequently, the forces constructed by inspection of these relationships and recorded as

$$\mathbf{C}_{is} = \lambda_s \mathbf{W}_{is} \qquad (i = 1, \ldots, \nu;\ s = 1, \ldots, M + m) \tag{44}$$

can be used together with any method of forming dynamical equations of motion that can deal with a force having a known direction and an unknown magnitude. Another advantage is that typically there is labor to be saved by writing equations in terms of vectors and dyadics rather than scalars. Moreover, the constraint forces can be identified selectively by forming only those equations that express the constraints that are of interest, and the multipliers are related unambiguously to the constraint forces.

2.3 Noncontributing Forces

The analysis performed thus far in this chapter deals with systems of individual particles; it is readily extended to accommodate systems in which rigid bodies comprise subsets of particles. There are, however, two important demonstrations that are easily made with the machinery already in hand. It is shown in Sec. 2.3.1 that the constraint forces formed by inspecting Eqs. (42) do not contribute to Kane's nonholonomic generalized active forces. In addition, Sec. 2.3.2 presents a demonstration that the constraint forces constructed upon inspection of Eqs. (39) make no contributions to Kane's holonomic generalized active forces (which include as a special case the generalized forces in Lagrange's equations for holonomic systems) or to nonholonomic generalized active forces in the event that motion constraints are also dictated.

2.3.1 Nonholonomic Constraint Forces

For the purpose of determining the contributions to Kane's generalized active forces from forces needed to constrain a system according to Eqs. (42), the relationships are brought into a more familiar scalar form by making use of Eq. (2.14.2) of Ref. [44] to express the velocity ${}^N\mathbf{v}^{P_i}$ of P_i in N in terms of holonomic partial velocities ${}^N\mathbf{v}_r^{P_i}$ of P_i in N.

$$ {}^N\mathbf{v}^{P_i} = \sum_{r=1}^{n} {}^N\mathbf{v}_r^{P_i}\, u_r + {}^N\mathbf{v}_t^{P_i} \qquad (i = 1, \ldots, \nu) \tag{45} $$

where ${}^N\mathbf{v}_r^{P_i}$ $(r = 1, \ldots, n)$ and ${}^N\mathbf{v}_t^{P_i}$ are vector functions of $q_1, \ldots, q_n$ and t in N; hence they have the same functional character as the vectors $\mathbf{W}_{is}$ in Eqs. (42). Substitution from Eqs. (45) into (42) gives

$$ \sum_{r=1}^{n} \left(\sum_{i=1}^{\nu} {}^N\mathbf{v}_r^{P_i} \cdot \mathbf{W}_{is} \right) u_r + \sum_{i=1}^{\nu} {}^N\mathbf{v}_t^{P_i} \cdot \mathbf{W}_{is} + Y_s = 0 \qquad (s = 1, \ldots, m) \tag{46} $$

The coefficients of u_r and the remaining terms can be abbreviated respectively by means of two definitions,

$$ \alpha_{sr} \triangleq \sum_{i=1}^{\nu} {}^N\mathbf{v}_r^{P_i} \cdot \mathbf{W}_{is} \qquad (s = 1, \ldots, m;\ r = 1, \ldots, n) \tag{47} $$

and

$$\beta_s \stackrel{\triangle}{=} Y_s + \sum_{i=1}^{\nu} {}^N\mathbf{v}_t^{P_i} \cdot \mathbf{W}_{is} \qquad (s = 1, \ldots, m) \tag{48}$$

These are general forms involving all particles belonging to S; on the other hand, Eqs. (6) in Ref. [27] represent a special case in which only two particles participate in the sums. The definitions allow Eqs. (46) to be rewritten in a form that is clearly linear in the motion variables, and that resembles Eqs. (10) in Ref. [79], Eqs. (22) in Ref. [78], Eqs. (3) in Ref. [9], and Eqs. (2) in Ref. [11]:

$$\sum_{r=1}^{n} \alpha_{sr} u_r + \beta_s = 0 \qquad (s = 1, \ldots, m) \tag{49}$$

These are the relationships written in matrix form in Secs. 1.1 and 1.4. In using Lagrange's method one is forced to adopt the special case of $u_r = \dot{q}_r$, and the associated literature is replete with the corresponding specialized nonholonomic form. Equations (49) express the dependency of m of the motion variables, say $u_{p+1}, \ldots, u_n$, on the remaining independent motion variables, say $u_1, \ldots, u_p$, where p, the number of degrees of freedom of S in N, has previously been defined as $p \stackrel{\triangle}{=} n - m$. The apparatus in Ref. [44] for dealing with nonholonomic systems rests on constraint equations that each relate one of the dependent motion variables to the independent motion variables. Such relationships are given by Eqs. (2.13.1) in Ref. [44] and can be written as

$$u_{p+r} = \sum_{s=1}^{p} A_{rs} u_s + B_r \qquad (r = 1, \ldots, m) \tag{50}$$

where we follow the practice adopted in Ref. [6] of shifting the index of the dependent motion variables so that the first rows of A and B are numbered 1 and the last rows are numbered m. When all nonholonomic constraint equations can be expressed with these relationships, S is said to be a simple nonholonomic system (Ref. [44], p. 43). To bring Eqs. (49) into this form, begin by putting Eqs. (50) in matrix form as in Eq. (5) of Ref. [6],

$$u_D = A u_I + B \tag{51}$$

where u_I is a $p\times 1$ column array containing the independent motion variables $u_1, \ldots, u_p$, u_D is an $m \times 1$ column array containing the dependent motion variables $u_{p+1}, \ldots, u_n$, A is an $m \times p$ matrix whose elements are A_{rs}, and B is an $m \times 1$ column array with elements B_r. Borrowing from the strategy of generalized coordinate partitioning (Refs. [66] and [18]), Eqs. (49) can be recast in matrix form with motion variable partitioning as

$$\alpha_I u_I + \alpha_D u_D + \beta = 0 \tag{52}$$

where α_I is an $m \times p$ matrix, α_D is an $m \times m$ matrix, and β is an $m \times 1$ column array whose elements are $\beta_1, \ldots, \beta_m$. As pointed out in Ref. [78], the motion variables can always be ordered such that α_D has an inverse as long as the constraint equations are independent, thus

$$u_D = -{\alpha_D}^{-1}\alpha_I u_I - {\alpha_D}^{-1}\beta \tag{53}$$

and comparison of this relationship with Eq. (51) produces the definitions

$$A \triangleq -{\alpha_D}^{-1}\alpha_I, \qquad B \triangleq -{\alpha_D}^{-1}\beta \tag{54}$$

In what follows, the first objective is to determine the contribution of the nonholonomic constraint forces $\mathbf{C}_{is}$ ($i = 1, \ldots, \nu; s = 1, \ldots, m$) to the holonomic generalized active forces F_r ($r = 1, \ldots, n$). The second goal is to show, in general, that $\mathbf{C}_{is}$ contribute nothing to the nonholonomic generalized active forces $\tilde{F}_r$ ($r = 1, \ldots, p$).

Holonomic generalized active forces for S in N are defined by Eqs. (4.4.2) in Ref. [44]:

$$F_r \triangleq \sum_{i=1}^{\nu} {}^N\mathbf{v}_r^{P_i} \cdot \mathbf{R}_i \qquad (r = 1, \ldots, n) \tag{55}$$

where, as stated previously, $\mathbf{R}_i = \mathbf{f}_i + \mathbf{C}_i$ is the resultant of all contact and distance forces acting on P_i, and ${}^N\mathbf{v}_r^{P_i}$ is the rth holonomic partial velocity of P_i in N. Thus $(F_r)_\mathcal{C}$, the contribution to F_r of the forces necessary to enforce Eqs. (43), is given by

$$(F_r)_\mathcal{C} = \sum_{i=1}^{\nu} {}^N\mathbf{v}_r^{P_i} \cdot \mathbf{C}_i \qquad (r = 1, \ldots, n) \tag{56}$$

where $\mathbf{C}_i$ is the resultant of the nonholonomic constraint forces applied to P_i, or, in view of Eqs. (44) with $M = 0$,

$$\mathbf{C}_i \triangleq \sum_{s=1}^{m} \mathbf{C}_{is} = \sum_{s=1}^{m} \lambda_s \mathbf{W}_{is} \qquad (i = 1, \ldots, \nu) \tag{57}$$

and, by reference to Eqs. (56), (57) and (47), the contribution is

$$(F_r)_\mathcal{C} = \sum_{i=1}^{\nu} {}^N\mathbf{v}_r^{P_i} \cdot \sum_{s=1}^{m} \lambda_s \mathbf{W}_{is} = \sum_{s=1}^{m} \lambda_s \alpha_{sr} \qquad (r = 1, \ldots, n) \tag{58}$$

It is worth emphasizing that $(F_r)_\mathcal{C}$ represents a contribution to F_r; the remainder of F_r consists of the contribution $(F_r)_\mathcal{F} \triangleq \sum_{i=1}^{\nu} {}^N\mathbf{v}_r^{P_i} \cdot \mathbf{f}_i$ of the forces acting on S when it is not restricted by the constraints under consideration. The final term $\sum_{s=1}^{m} \lambda_s \alpha_{sr}$ in Eqs. (58) is analagous to the third term in the left hand member of Eqs. (11) in Ref. [27]; however, a conflict exists in the discussion of Eqs. (2) and (11) in that work. The F_r in Eqs. (2) are said to be formed from the resultant of *all* active forces acting on P_i, which must include the constraint forces. This amounts to saying the constraint forces contribute twice to Eqs. (11); once to the first term F_r, and again to the third term $\sum_{k=p+1}^{n} A_{kr} R_k$. Of course, the constraint force contributions can only be counted once. In contrast, the discussion of Eqs. (4) in Ref. [9] and Eqs. (3) in Ref. [11] correctly describes their use of F_r to denote forces *other than* the constraint forces, and the final sum in Eqs. (58) here is hence identical to the sum appearing in the aforementioned two relationships.

When $u_r \triangleq \dot{q}_r$, $(F_r)_\mathcal{C}$ becomes the generalized force needed to enforce a constraint in Lagrange's equations for a nonholonomic system, denoted by $Q_r{}'$ on p. 215 of Ref. [82]. As mentioned in Sec. 1.3, one of the objectives of Ref. [37] is to establish the validity of $(F_r)_\mathcal{C} = \sum_{s=1}^{m} \lambda_s \alpha_{sr}$. This is accomplished by choosing $u_r \triangleq \dot{q}_r$ and prescribing the motion of a single member of a system of rigid bodies. A similar exercise in Ref. [79] also is based on prescribed motion, although the number of points and bodies involved is arbitrary, and u_r is not limited to $\dot{q}_r$. A motion constraint, such as rolling, is not used to construct either of the proofs. In contrast, the result obtained here is general on three counts. First, it holds for any valid choice of motion variables, second, it involves all of the particles belonging to a system (whether or not some of the particles make up rigid bodies), and third, it is true for any system subject to motion constraints described by nonholonomic equations that are linear in the motion variables.

Equations (58) can be expressed in matrix form as

$$(F)_{\mathcal{C}} = \alpha^{\mathrm{T}}\lambda = \begin{Bmatrix} {\alpha_I}^{\mathrm{T}}\lambda \\ {\alpha_D}^{\mathrm{T}}\lambda \end{Bmatrix} \triangleq \begin{Bmatrix} (F_I)_{\mathcal{C}} \\ (F_D)_{\mathcal{C}} \end{Bmatrix} \tag{59}$$

where λ is an $m \times 1$ column array whose elements are $\lambda_1, \ldots, \lambda_m$, $(F_I)_{\mathcal{C}}$ is a $p \times 1$ column array with elements $(F_1)_{\mathcal{C}}, \ldots, (F_p)_{\mathcal{C}}$, and $(F_D)_{\mathcal{C}}$ is an $m \times 1$ column array with elements $(F_{p+1})_{\mathcal{C}}, \ldots, (F_n)_{\mathcal{C}}$.

Nonholonomic generalized active forces $\tilde{F}_r$ for S in N are used to form a minimal set of dynamical equations and are given in terms of the holonomic generalized active forces F_r by Eqs. (4.4.3) in Ref. [44]. Application of those relationships to the constraint force contributions yields

$$(\tilde{F}_r)_{\mathcal{C}} = (F_r)_{\mathcal{C}} + \sum_{s=1}^{m} (F_{p+s})_{\mathcal{C}}\ A_{sr} \qquad (r = 1, \ldots, p) \tag{60}$$

where, once again, indices are shifted so that the first row of the matrix A is numbered 1 rather than $p + 1$. These relationships can be expressed in matrix form as

$$\begin{aligned}(\tilde{F})_{\mathcal{C}} &= (F_I)_{\mathcal{C}} + A^{\mathrm{T}}(F_D)_{\mathcal{C}} \\ &= {\alpha_I}^{\mathrm{T}}\lambda + A^{\mathrm{T}}{\alpha_D}^{\mathrm{T}}\lambda = \left({\alpha_I}^{\mathrm{T}} + A^{\mathrm{T}}{\alpha_D}^{\mathrm{T}}\right)\lambda \end{aligned} \tag{61}$$

The term in parentheses is observed to vanish by noting

$$0 = \alpha_I - \alpha_I = \alpha_I - \alpha_D {\alpha_D}^{-1}\alpha_I = \alpha_I + \alpha_D A \tag{62}$$

Hence, the transpose of this relationship is ${\alpha_I}^{\mathrm{T}} + A^{\mathrm{T}}{\alpha_D}^{\mathrm{T}} = 0$. This step may be viewed as premultiplication of α^{T} by an orthogonal complement, the matrix denoted by $A_2 = [I_p \; A^{\mathrm{T}}]$ in Ref. [6], where I_p is the $p \times p$ identity matrix. In any event, it is shown that

$$(\tilde{F}_r)_{\mathcal{C}} = 0 \qquad (r = 1, \ldots, p) \tag{63}$$

or, in words, motion constraints expressible in the form of Eqs. (42) require the application of forces that make no contributions to any of the nonholonomic generalized active forces. Such a demonstration is alluded to in Ref. [11]. Because these contributions are defined in terms of nonholonomic partial velocities ${}^N\tilde{\mathbf{v}}_r^{P_i}$ after referring to Eqs. (4.4.1) of Ref. [44]

$$(\tilde{F}_r)_{\mathcal{C}} = \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{C}_i = \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \sum_{s=1}^{m} \lambda_s \mathbf{W}_{is} = \sum_{s=1}^{m} \lambda_s \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{W}_{is} \qquad (r = 1, \ldots, p) \tag{64}$$

it can be concluded that

$$\sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{W}_{is} = 0 \qquad (r = 1, \ldots, p;\; s = 1, \ldots, m) \tag{65}$$

2.3.2 Holonomic Constraint Forces

The foregoing analysis shows the forces needed to ensure adherence to motion constraints contribute nothing to the nonholonomic generalized active forces; with similar reasoning one can establish that forces needed to ensure satisfaction of configuration constraints contribute nothing to the holonomic generalized active forces, and therefore they contribute nothing to the nonholonomic generalized active forces should the system be subject to motion constraints as well as configuration constraints.

To see this, assume that the configuration of the unconstrained system is described by $n + M$ generalized coordinates and that the motion is characterized by $n + M$ generalized speeds. The M additional coordinates and speeds are referred to as pseudo-generalized coordinates and pseudo-generalized speeds (Ref. [44], p. 300). In Refs. [25] and [67] a rigid holonomic multibody system used to model the biomechanics of human motion is regarded first as unconstrained and described by $n + M$

natural coordinates, proposed in Ref. [30], related to one another by M equations of constraint. A non-independent set of reference point coordinates is used in a similar manner in the analysis of holonomic multibody systems, as discussed in Ref. [74].

Configuration constraints can be expressed in vector form as Eqs. (37), and these can be differentiated twice with respect to time in N to arrive at the M Eqs. (41). In what was done in Sec. 2.3.1, replace all index limits of m, p, and n with M, n, and $n+M$ respectively. There will then be $n+M$ unconstrained partial velocities and $n+M$ contributions of the constraint forces to unconstrained generalized active forces rather than n as in Eqs. (56) and (58). A counterpart of Eq. (60) relates n contributions to holonomic generalized active forces on the left to $n+M$ contributions to unconstrained generalized active forces on the right, and a counterpart to Eq. (63) then shows that there are no contributions $(F_r)_{\mathcal{C}}$ to the holonomic generalized active forces $(r = 1, \ldots, n)$ from the forces needed to ensure the configuration constraints are obeyed. For holonomic systems with $u_r \triangleq \dot{q}_r$, the generalized active force F_r is identical (Ref. [44], pp. 327–328) to the generalized force in Lagrange's equations of the first kind, often denoted by Q_r. Therefore, this result applies both to Kane's equations and to Lagrange's equations. The conclusion expressed in Eq. (65) gives way to

$$\sum_{i=1}^{\nu} {}^{N}\mathbf{v}_r^{P_i} \cdot \mathbf{W}_{is} = 0 \qquad (r = 1, \ldots, n;\ s = 1, \ldots, M) \tag{66}$$

for the vectors $\mathbf{W}_{is}$ appearing in the holonomic constraint equations (39).

Finally, use of Eqs. (60) as they appear reveals that $(\tilde{F}_r)_{\mathcal{C}}$ must all vanish $(r = 1, \ldots, p)$ because all of $(F_r)_{\mathcal{C}}$ due to configuration constraints are zero $(r = 1, \ldots, n)$. Thus, the forces needed to constrain the positions of particles in a system contribute nothing to the nonholonomic generalized active forces in the event the system is subject both to M configuration constraints and m motion constraints.

2.4 Constraint Forces Acting on a Rigid Body

When a subset of particles $P_1, \ldots, P_\beta$ belonging to a system S form a rigid body B, nonholonomic constraint equations (42) can be written in terms of the velocity ${}^N\mathbf{v}^Q$ in N of a point Q fixed in B, and the angular velocity ${}^N\boldsymbol{\omega}^B$ of B in N.

$$\begin{aligned}\sum_{i=1}^{\beta} {}^N\mathbf{v}^{P_i} \cdot \mathbf{W}_{is} &= \sum_{i=1}^{\beta}({}^N\mathbf{v}^Q + {}^N\boldsymbol{\omega}^B \times \mathbf{r}_i) \cdot \mathbf{W}_{is} \\ &= {}^N\mathbf{v}^Q \cdot \sum_{i=1}^{\beta}\mathbf{W}_{is} + {}^N\boldsymbol{\omega}^B \cdot \sum_{i=1}^{\beta}\mathbf{r}_i \times \mathbf{W}_{is} \\ &\triangleq {}^N\mathbf{v}^Q \cdot \mathbf{W}_s + {}^N\boldsymbol{\omega}^B \cdot \boldsymbol{\tau}_s \qquad (s = 1, \ldots, m)\end{aligned} \tag{67}$$

where $\mathbf{r}_i$ is the position vector from Q to P_i.

The material in Sec. 2.2.2 establishes that the appearance of the vector $\mathbf{W}_{is}$ in Eqs. (67) requires the application of a constraint force $\mathbf{C}_{is} = \lambda_s \mathbf{W}_{is}$ to P_i. After selecting the line of action of $\mathbf{W}_{is}$ such that it passes through P_i, and defining the resultants

$$\mathbf{W}_s \triangleq \sum_{i=1}^{\beta}\mathbf{W}_{is}, \qquad \mathbf{C}_s \triangleq \sum_{i=1}^{\beta}\mathbf{C}_{is} \qquad (s = 1, \ldots, m) \tag{68}$$

the set of forces $\mathbf{C}_{1s}, \ldots, \mathbf{C}_{\beta s}$ applied to B is regarded as equivalent to a single force $\mathbf{C}_s$ whose line of action passes through Q, together with a couple whose torque is $\mathbf{T}_s$. The resultant $\mathbf{C}_s$ is given by

$$\mathbf{C}_s = \sum_{i=1}^{\beta}\mathbf{C}_{is} = \sum_{i=1}^{\beta}\lambda_s\mathbf{W}_{is} = \lambda_s\mathbf{W}_s \qquad (s = 1, \ldots, m) \tag{69}$$

and the torque $\mathbf{T}_s$ is equal to the moment of $\mathbf{C}_{1s}, \ldots, \mathbf{C}_{\beta s}$ about Q,

$$\mathbf{T}_s = \sum_{i=1}^{\beta}\mathbf{r}_i \times \mathbf{C}_{is} = \sum_{i=1}^{\beta}\mathbf{r}_i \times \lambda_s\mathbf{W}_{is} = \lambda_s\boldsymbol{\tau}_s \qquad (s = 1, \ldots, m) \tag{70}$$

where $\boldsymbol{\tau}_s$ is the moment of $\mathbf{W}_{1s}, \ldots, \mathbf{W}_{\beta s}$ about Q,

$$\boldsymbol{\tau}_s \triangleq \sum_{i=1}^{\beta}\mathbf{r}_i \times \mathbf{W}_{is} \qquad (s = 1, \ldots, m) \tag{71}$$

One is therefore in a position to say that the appearance of the dot product ${}^N\mathbf{v}^Q \cdot \mathbf{W}_s$ requires that B is subject to a constraint force $\mathbf{C}_s = \lambda_s\mathbf{W}_s$ applied to

Q, and the appearance of the dot product ${}^N\boldsymbol{\omega}^B \cdot \boldsymbol{\tau}_s$ means B must be acted upon by a couple whose constraint torque is $\mathbf{T}_s = \lambda_s \boldsymbol{\tau}_s$ ($s = 1, \ldots, m$). This result is the insightful observation made in Ref. [77], that inspection of a constraint equation containing the dot products ${}^N\mathbf{v}^Q \cdot \mathbf{W}_s$ and ${}^N\boldsymbol{\omega}^B \cdot \boldsymbol{\tau}_s$ indicates the direction of a constraint force and the point to which it must be applied, together with the direction of a constraint torque and the body upon which it must be exerted. The analysis presented in this chapter leading up to this conclusion provides a rigorous justification for such an observation. It is worth noting that the result depends upon the participation in Eqs. (42) of all particles belonging to S and, consequently, all of the particles that constitute a rigid body.

Just as a rigid body makes a contribution $(F_r^\star)_B$ to each holonomic generalized inertia force, and the set of forces $\mathbf{f}_i$ acting on B contributes $(F_r)_B$ to the rth holonomic generalized active force for S in N (Ref. [44], pp. 106, 124, and 125, with $p = n$), the constraint forces applied to B to ensure adherence to all m nonholonomic relationships make a contribution to the holonomic generalized active force given by

$$\begin{aligned}[(F_r)_{\mathcal{C}}]_B &= {}^N\mathbf{v}_r^Q \cdot \sum_{s=1}^m \mathbf{C}_s + {}^N\boldsymbol{\omega}_r^B \cdot \sum_{s=1}^m \mathbf{T}_s = {}^N\mathbf{v}_r^Q \cdot \sum_{s=1}^m \lambda_s \mathbf{W}_s + {}^N\boldsymbol{\omega}_r^B \cdot \sum_{s=1}^m \lambda_s \boldsymbol{\tau}_s \\ &= \sum_{s=1}^m \lambda_s (\alpha_{sr})_B \qquad (r = 1, \ldots, n) \end{aligned} \tag{72}$$

where the contribution of B to each element of the matrix α is defined as

$$(\alpha_{sr})_B \triangleq {}^N\mathbf{v}_r^Q \cdot \mathbf{W}_s + {}^N\boldsymbol{\omega}_r^B \cdot \boldsymbol{\tau}_s \qquad (s = 1, \ldots, m; r = 1, \ldots, n) \tag{73}$$

Because Eqs. (39) have the same form as Eqs. (42), holonomic constraint relationships expressed at the velocity level can likewise be inspected to determine the point of application and direction of a constraint force together with the body of application and the direction of a constraint torque needed to enforce a configuration constraint. The contribution to certain generalized active forces from the set of holonomic constraint forces applied to a rigid body is determined from Eqs. (72) and (73) with index limits of m and n replaced by M, and $n + M$, respectively.

2.5 Introducing Additional Motion Variables

Constraint forces obtained with the approach taken in Sec. 2.2 are seen in Sec. 2.3 to be noncontributing forces. Therefore, it is not surprising that there exists a close correspondence between bringing constraint forces into the picture by inspection as described in Secs. 2.2 and 2.4 on the one hand, and bringing noncontributing forces into evidence by introducing additional motion variables on the other. The following material illustrates that the present method lends itself well to the practice established in Ref. [44]. A natural fit is shown by example in Sec. 2.5.1 subsequent to a review of the procedure for bringing noncontributing forces into evidence in a simple nonholonomic system, and the similar procedure for a holonomic system. Two strategies for introducing additional motion variables are compared in Sec. 2.5.2, and the question of which one might be preferable is entertained. This is followed in Sec. 2.5.3 with brief comments regarding the versatility of the method for bringing constraint forces to light.

2.5.1 The Generic Method

According to p. 114 of Ref. [44], "The introduction of a suitable additional generalized speed is accomplished by permitting points to have certain *velocities*, or rigid bodies to have certain *angular velocities*, which they cannot, in fact possess…" Emphasis has been added to draw attention to the mention of vector quantities; this is tantamount to saying that the velocity ${}^N\mathbf{v}^{P_2}$ may differ from the velocity ${}^N\mathbf{v}^{P_1}$, or that the angular velocity ${}^N\boldsymbol{\omega}^B$ may differ from the angular velocity ${}^N\boldsymbol{\omega}^A$, when in fact it cannot. Introduction of an additional motion variable (generalized speed) is in effect the introduction of a constraint equation. It is worth reviewing the established procedure for bringing noncontributing forces into evidence, first by limiting the discussion to a simple nonholonomic system. Suitable adjustments to the discussion then make it applicable to a holonomic system.

In the case of a simple nonholonomic system, allow the motion variables $u_1, \ldots, u_p$ to characterize motion that the system can in fact undergo, and limit the additional motion variables to be $u_{p+1}, \ldots, u_n$, corresponding to velocities that particles cannot in fact possess. In the following description it is assumed that one has first formed the equations of motion $\tilde{F}_r + \tilde{F}_r^\star = 0$ $(r = 1, \ldots, p)$ that lack evidence of noncontributing forces, and one subsequently wishes to have them appear in new equations of motion.

The first step is to introduce additional motion variables $u_{p+1}, \ldots, u_n$ into the analysis (assuming one is interested in the noncontributing forces associated with all m nonholonomic constraint equations). Next, one obtains additional partial velocities ${}^N\mathbf{v}_{p+s}^{P_i}$ $(i = 1, \ldots, \nu; s = 1, \ldots, m)$ corresponding to the additional motion variables. Once this has been done, u_{p+s} must be set to their actual values as indicated on p. 125 of Ref. [44]. This is normally accomplished by employing scalar constraint relationships in the form of Eqs. (50); however, in view of the developments in Sec. 2.3, one may instead appeal to Eqs. (42) that contain velocity vectors explicitly. The additional partial velocities are used to obtain additional generalized active forces F_{p+s} and additional generalized inertia forces $F_{p+s}^\star$. The quantities ${}^N\tilde{\mathbf{v}}_r^{P_i}$, $\tilde{F}_r$, and $\tilde{F}_r^\star$ corresponding to the original motion variables $u_1, \ldots, u_p$ (and, hence, to the actual motion the system can have) are formed prior to introducing the additional motion variables. Because the motion constraints expressed by Eqs. (50) relate the additional motion variables characterizing the fictitious motion to the original motion variables describing the actual motion, Eqs. (2.14.17), (4.4.3), and (4.11.4) of Ref. [44] are in effect. Therefore, the quantities ${}^N\mathbf{v}_r^{P_i}$, F_r, and $F_r^\star$ corresponding to the original motion variables, and formed subsequent to introducing the additional motion variables, are given by

$${}^N\mathbf{v}_r^{P_i} = {}^N\tilde{\mathbf{v}}_r^{P_i} - \sum_{s=1}^{m} {}^N\mathbf{v}_{p+s}^{P_i} A_{sr} \qquad (r = 1, \ldots, p) \tag{74}$$

$$F_r = \tilde{F}_r - \sum_{s=1}^{m} F_{p+s} A_{sr} \qquad (r = 1, \ldots, p) \tag{75}$$

$$F_r^\star = \tilde{F}_r^\star - \sum_{s=1}^{m} F_{p+s}^\star A_{sr} \qquad (r = 1, \ldots, p) \tag{76}$$

In general the left hand members of these relationships differ from ${}^N\tilde{\mathbf{v}}_r^{P_i}$, $\tilde{F}_r$, and $\tilde{F}_r^\star$ when any of A_{sr} are nonzero $(s = 1, \ldots, m)$. The equations in which the noncontributing forces are in evidence are given by $F_r + F_r^\star = 0$, $(r = 1, \ldots, n)$. It has been demonstrated in Sec. 2.3.1 that the forces $\mathbf{C}_i$ needed to enforce motion constraints contribute $(F_r)_\mathcal{C}$ to F_r according to Eqs. (58), even though $\mathbf{C}_i$ $(i = 1, \ldots, \nu)$ are not in evidence in $\tilde{F}_r$ $(r = 1, \ldots, p)$.

The introduction of additional motion variables (pseudo-generalized speeds) to bring noncontributing forces into evidence in a holonomic system can be regarded in a manner similar to the preceding treatment of a nonholonomic system. In general the same comments apply with all index limits of m, p, and n replaced by M, n, and $n + M$ respectively. However, instead of indicating nonholonomic quantities, the tildes in Eqs. (74)–(76) indicate holonomic quantities corresponding to the original motion variables $u_1, \ldots, u_n$ and constructed prior to the introduction of the additional motion variables. The quantities in the sums, ${}^N\mathbf{v}_{n+s}^{P_i}$, F_{n+s}, and $F_{n+s}^\star$, are associated with the additional motion variables u_{n+s} $(s = 1, \ldots, M)$. The left hand members ${}^N\mathbf{v}_r^{P_i}$, F_r, and $F_r^\star$ $(r = 1, \ldots, n)$ then represent holonomic quantities corresponding to the original motion variables and constructed subsequent to the introduction of the additional motion variables, and the equations in which the noncontributing forces are in evidence are given by $F_r + F_r^\star = 0$, $(r = 1, \ldots, n + M)$.

Example 1 Consider a rigid body B moving in a Newtonian reference frame N, as shown in Figure 4. A particle P belongs to B and remains coincident with a point $\overline{B}$ of B. If B belongs to a spacecraft in orbit and P is a particle that is part of a science experiment, the investigator is often greatly interested in the constraint force exerted by B and P on each other in order to keep P fixed in B. The constraint force per unit mass, which can be measured by an

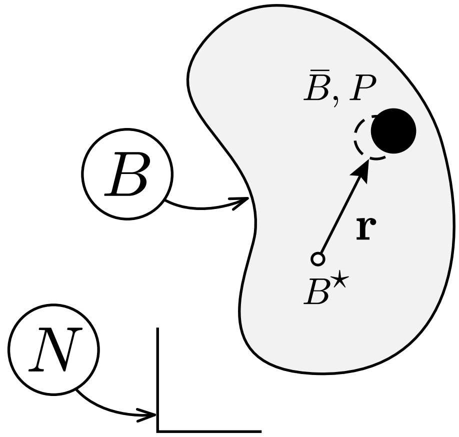


Figure 4: Particle P Fixed in Body B

accelerometer, is typically referred to as the level of microgravity. (Unfortunately, this terminology frequently leads to the mistaken belief that the force of gravity acting on an object in orbit is but a minute fraction of that acting on the object when it is near Earth's surface.)

Certain information about the constraint force can be obtained as described in Sec. 2.4. When P is constrained to be fixed in B, ${}^N\mathbf{v}^P$ must be identical to ${}^N\mathbf{v}^{\overline{B}}$. Three holonomic constraint equations can be written at the velocity level as

$$ {}^N\mathbf{v}^P \cdot \hat{\mathbf{b}}_s + {}^N\mathbf{v}^{\overline{B}} \cdot (-\hat{\mathbf{b}}_s) = 0 \qquad (s = 1, 2, 3) \tag{77} $$

where $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, and $\hat{\mathbf{b}}_3$ are right-handed, mutually orthogonal unit vectors fixed in B. The constraint force exerted by B on P is then given by inspection as

$$ \mathbf{C} = \lambda_1\hat{\mathbf{b}}_1 + \lambda_2\hat{\mathbf{b}}_2 + \lambda_3\hat{\mathbf{b}}_3 \tag{78} $$

The constraint force exerted by P on B, likewise obtained by inspection, is simply $-\mathbf{C}$ applied at $\overline{B}$. Equations (77) can be regarded as the introduction of an additional velocity rather than additional motion variables, and the process of inspection brings $\mathbf{C}$ and $-\mathbf{C}$ into evidence in vector form without the explicit introduction of additional motion variables.

The constraint force $\mathbf{C}$ is noncontributing; however, its measure numbers can be brought into evidence in generalized active forces, and hence in equations of motion, according to the procedures established in Ref. [44]. Define motion variables $u_1, \ldots, u_6$ operationally as

$$ {}^{N}\mathbf{v}^{B^\star} = u_1\hat{\mathbf{b}}_1 + u_2\hat{\mathbf{b}}_2 + u_3\hat{\mathbf{b}}_3, \qquad {}^{N}\boldsymbol{\omega}^{B} = u_4\hat{\mathbf{b}}_1 + u_5\hat{\mathbf{b}}_2 + u_6\hat{\mathbf{b}}_3 \tag{79} $$

where $B^\star$ is the mass center of B. If $\mathbf{r}$ is the position vector from $B^\star$ to $\overline{B}$, then the velocity of $\overline{B}$ in N is given by

$$ {}^{N}\mathbf{v}^{\overline{B}} = {}^{N}\mathbf{v}^{B^\star} + {}^{N}\boldsymbol{\omega}^{B} \times \mathbf{r} \tag{80} $$

The partial velocities of $\overline{B}$ in N corresponding to the original motion variables are

$$ {}^{N}\mathbf{v}_r^{\overline{B}} = \hat{\mathbf{b}}_r \qquad (r = 1, 2, 3), \qquad {}^{N}\mathbf{v}_r^{\overline{B}} = \hat{\mathbf{b}}_{r-3} \times \mathbf{r} \qquad (r = 4, 5, 6) \tag{81} $$

The contribution of $\mathbf{C}$ to F_r associated with the original six motion variables vanishes, of course,

$$ (F_r)_{\mathcal{C}} = {}^{N}\mathbf{v}_r^{P} \cdot \mathbf{C} + {}^{N}\mathbf{v}_r^{\overline{B}} \cdot (-\mathbf{C}) = 0 \qquad (r = 1, \ldots, 6) \tag{82} $$

because all of ${}^{N}\mathbf{v}_r^{P}$ are in fact identical to ${}^{N}\mathbf{v}_r^{\overline{B}}$. For the purpose of bringing $\mathbf{C}$ into evidence, P is allowed to have a velocity different than that of $\overline{B}$ and ${}^{N}\mathbf{v}^{P}$ is expressed in terms of three additional motion variables; one way of introducing them is to write

$$ {}^{N}\mathbf{v}^{P} = u_7\hat{\mathbf{b}}_1 + u_8\hat{\mathbf{b}}_2 + u_9\hat{\mathbf{b}}_3 \tag{83} $$

Partial velocities associated with the additional motion variables are given by

$$ {}^{N}\mathbf{v}_r^{\overline{B}} = \mathbf{0}, \qquad {}^{N}\mathbf{v}_r^{P} = \hat{\mathbf{b}}_{r-6} \qquad\qquad (r = 7, 8, 9) \tag{84} $$

The constraint equations (77) can be manipulated to express the additional motion variables in terms of the original motion variables,

$$u_7 = u_1 + u_5 r_3 - u_6 r_2 \tag{85}$$

$$u_8 = u_2 - u_4 r_3 + u_6 r_1 \tag{86}$$

$$u_9 = u_3 + u_4 r_2 - u_5 r_1 \tag{87}$$

where $r_i \triangleq \mathbf{r} \cdot \hat{\mathbf{b}}_i$, $(i = 1, 2, 3)$. These relationships have the form of Eqs. (50); the nonzero values of A_{rs} are identified by inspection,

$$A_{11} = 1, \qquad A_{15} = r_3, \qquad A_{16} = -r_2 \tag{88}$$

$$A_{22} = 1, \qquad A_{24} = -r_3, \qquad A_{26} = r_1 \tag{89}$$

$$A_{33} = 1, \qquad A_{34} = r_2, \qquad A_{35} = -r_1 \tag{90}$$

and used to evaluate Eqs. (74) for the holonomic system at points $\overline{B}$ and P. Because Eqs. (84) show that ${}^N\mathbf{v}^{\overline{B}}_{6+s} = \mathbf{0}$ $(s = 1, 2, 3)$, the partial velocities of $\overline{B}$ associated with $u_1, \ldots, u_6$ subsequent to the introduction of u_7, u_8, and u_9 are unchanged and therefore given by Eqs. (81). The partial velocities of P, equal to the partial velocities of $\overline{B}$ prior to introducing additional motion variables, are found to be subsequently altered.

$${}^N\mathbf{v}^P_1 = \hat{\mathbf{b}}_1 - \hat{\mathbf{b}}_1(1) - \hat{\mathbf{b}}_2(0) - \hat{\mathbf{b}}_3(0) = \mathbf{0} \tag{91}$$

$${}^N\mathbf{v}^P_2 = \hat{\mathbf{b}}_2 - \hat{\mathbf{b}}_1(0) - \hat{\mathbf{b}}_2(1) - \hat{\mathbf{b}}_3(0) = \mathbf{0} \tag{92}$$

$${}^N\mathbf{v}^P_3 = \hat{\mathbf{b}}_3 - \hat{\mathbf{b}}_1(0) - \hat{\mathbf{b}}_2(0) - \hat{\mathbf{b}}_3(1) = \mathbf{0} \tag{93}$$

$${}^N\mathbf{v}^P_4 = \hat{\mathbf{b}}_1 \times \mathbf{r} - \hat{\mathbf{b}}_1(0) - \hat{\mathbf{b}}_2(-r_3) - \hat{\mathbf{b}}_3(r_2) = \mathbf{0} \tag{94}$$

$${}^N\mathbf{v}^P_5 = \hat{\mathbf{b}}_2 \times \mathbf{r} - \hat{\mathbf{b}}_1(r_3) - \hat{\mathbf{b}}_2(0) - \hat{\mathbf{b}}_3(-r_1) = \mathbf{0} \tag{95}$$

$${}^N\mathbf{v}^P_6 = \hat{\mathbf{b}}_3 \times \mathbf{r} - \hat{\mathbf{b}}_1(-r_2) - \hat{\mathbf{b}}_2(r_1) - \hat{\mathbf{b}}_3(0) = \mathbf{0} \tag{96}$$

The constraint forces exerted by B and P on each other make contributions

$(F_r)_{\mathcal{C}}$ to the generalized active forces

$$(F_r)_{\mathcal{C}} = {}^N\mathbf{v}_r^P \cdot \mathbf{C} + {}^N\mathbf{v}_r^{\overline{B}} \cdot (-\mathbf{C}) \qquad (r = 1, \ldots, 9) \tag{97}$$

or

$$(F_r)_{\mathcal{C}} = (\mathbf{0} - \hat{\mathbf{b}}_r) \cdot \mathbf{C} = -\lambda_r \qquad (r = 1, 2, 3) \tag{98}$$

$$(F_4)_{\mathcal{C}} = (\mathbf{0} - \hat{\mathbf{b}}_1 \times \mathbf{r}) \cdot \mathbf{C} = r_3\lambda_2 - r_2\lambda_3 \tag{99}$$

$$(F_5)_{\mathcal{C}} = (\mathbf{0} - \hat{\mathbf{b}}_2 \times \mathbf{r}) \cdot \mathbf{C} = r_1\lambda_3 - r_3\lambda_1 \tag{100}$$

$$(F_6)_{\mathcal{C}} = (\mathbf{0} - \hat{\mathbf{b}}_3 \times \mathbf{r}) \cdot \mathbf{C} = r_2\lambda_1 - r_1\lambda_2 \tag{101}$$

$$(F_r)_{\mathcal{C}} = (\hat{\mathbf{b}}_{r-6} - \mathbf{0}) \cdot \mathbf{C} = \lambda_{r-6} \qquad (r = 7, 8, 9) \tag{102}$$

The first three contributions are recognized as measure numbers of the constraint force $-\mathbf{C}$ exerted by P on $\overline{B}$, the final three contributions are measure numbers of the constraint force $\mathbf{C}$ exerted by B on P, and the remaining contributions are measure numbers of the moment of $-\mathbf{C}$ about $B^\star$, where all measure numbers are for the basis formed by $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, and $\hat{\mathbf{b}}_3$. One may verify that $(F_r)_{\mathcal{C}} + \sum_{s=1}^{3}(F_{6+s})_{\mathcal{C}} A_{sr} = 0$ for $r = 1, \ldots, 6$.

2.5.2 A Comparison of Methods

Two aspects of the foregoing example deserve further discussion. First, the contributions $(F_r)_{\mathcal{C}}$ associated with the original motion variables $u_1, \ldots, u_6$ have to be recalculated subsequent to introducing the additional motion variables u_7, u_8, and u_9. To start with they are given by $(F_r)_{\mathcal{C}} = 0$ $(r = 1, \ldots, 6)$ in Eqs. (82), whereas the subsequent revised expressions are reported in Eqs. (98)–(101). Second, more than one multiplier appears in each of Eqs. (99)–(101); solving the equations of motion for the multipliers would be simplified if each contribution $(F_r)_{\mathcal{C}}$ contained at most only one multiplier. A different choice of additional motion variables, to be presented shortly, allows one to avoid the labor required to revise $(F_1)_{\mathcal{C}}, \ldots, (F_6)_{\mathcal{C}}$. At the same time, use of these alternate variables ensures that at most one multiplier appears in each contribution $(F_r)_{\mathcal{C}}$ $(r = 1, \ldots, 9)$; moreover, such appearances are restricted to $(F_7)_{\mathcal{C}}$, $(F_8)_{\mathcal{C}}$, and $(F_9)_{\mathcal{C}}$.

It has been mentioned in connection with Eqs. (74) and (75) that generalized active forces formed prior to the introduction of additional motion variables remain subsequently unaltered when all of A_{sr} are zero. As shown in Ref. [27], introduction of additional motion variables that vanish during constrained motion (in which case all of A_{sr} are zero) has the advantageous consequence wherein evidence of the constraint forces is kept out of the original generalized active forces and thus is restricted to appear in the additional generalized active forces associated with the additional motion variables. Furthermore, Djerassi demonstrates that only one measure number of a constraint force or torque (that is, one multiplier) shows up in each additional generalized active force. This result is confirmed by Eqs. (59) with α_I equal to an $m \times p$ matrix of zeros and α_D equal to the $m \times m$ identity matrix.

Example 2 In Example 1, additional motion variables that have actual values of zero when P remains coincident with $\overline{B}$ can be introduced by writing

$$^{B}\mathbf{v}^{P} = u_7\hat{\mathbf{b}}_1 + u_8\hat{\mathbf{b}}_2 + u_9\hat{\mathbf{b}}_3 \tag{103}$$

in place of Eq. (83), where $^{B}\mathbf{v}^{P}$ is the velocity of P in B, and $^{N}\mathbf{v}^{P} = {}^{N}\mathbf{v}^{\overline{B}} + {}^{B}\mathbf{v}^{P}$. The partial velocities associated with the additional motion variables, $^{N}\mathbf{v}_r^{\overline{B}}$ and $^{N}\mathbf{v}_r^{P}$ $(r = 7, 8, 9)$, are once again given by Eqs. (84). Equations (85)–(87) are replaced by

$$u_r = 0 \qquad (r = 7, 8, 9) \tag{104}$$

In view of Eqs. (50), all of A_{sr} are 0 $(s = 1, 2, 3; r = 1, \ldots, 6)$. Regarded in another way with reference to Eqs. (52), α_I is a 3×6 matrix of zeros, α_D is the 3×3 identity matrix, and β is a 3×1 matrix of zeros. Thus, by choosing additional motion variables in this way, Eqs. (82) remain in effect and take the place of Eqs. (98)–(101), whereas Eqs. (102) continue to apply.

$$(F_r)_{\mathcal{C}} = 0 \qquad (r = 1, \ldots, 6) \tag{105}$$

$$(F_r)_{\mathcal{C}} = \lambda_{r-6} \qquad (r = 7, 8, 9) \tag{106}$$

The example illustrates that the method given in Sec. 2.4 for identifying directions, and points or bodies of application of constraint forces or torques, is completely unaffected by the choice of motion variables (or, for that matter, the choice of method for formulating equations of motion) because one works with velocities of points and angular velocities of rigid bodies, which are vectors. On the other hand, the process of bringing constraint forces and torques into evidence in Kane's equations is very much affected by the choice of motion variables. Mitiguy and Kane present in Ref. [52] a choice of motion variables for characterizing rotational motion so that dynamical equations of motion are simplified, in some cases strikingly, in comparison to those obtained with a customary choice. In Ref. [10] Banerjee expands upon the idea to present a choice of motion variable for a prismatic joint. Use of Banerjee's recommendation in forming additional motion variables characterizing fictitious translational motion of P in B leads to Eq. (83), whereas introducing them as in Eq. (103) could be considered a customary choice. The conclusions drawn from this example are logically extended to the general case. Solving for a constraint force measure number is made easier when the additional motion variable introduced is a customary one, chosen such that there is no relative motion between two bodies when the speed goes to zero. Although the motion variables proposed in Refs. [52] and [10] are not zero when relative motion is prevented, such as by a joint, their aforementioned advantages may nevertheless outweigh the increased difficulty in solving for constraint force measure numbers. The matter of choosing additional motion variables, therefore, deserves further study in future work.

2.5.3 Convenience and Selectivity

If, in Example 1, one is interested in the values of $\mathbf{C} \cdot \hat{\mathbf{a}}_s$ rather than $\mathbf{C} \cdot \hat{\mathbf{b}}_s$, where unit vectors $\hat{\mathbf{a}}_s$ $(s = 1, 2, 3)$ are fixed in some reference frame A that differs from B, then one needs only to write Eqs. (77) in terms of $\hat{\mathbf{a}}_s$ rather than $\hat{\mathbf{b}}_s$. A process of inspection is performed as indicated in Sec. 2.4 to obtain $\mathbf{C} = \mu_1\hat{\mathbf{a}}_1+\mu_2\hat{\mathbf{a}}_2+\mu_3\hat{\mathbf{a}}_3$ in place of Eq. (78). Likewise, in Example 2, μ_1, μ_2, and μ_3 are brought into evidence in $(F_7)_\mathcal{C}$, $(F_8)_\mathcal{C}$, and $(F_9)_\mathcal{C}$ by introducing additional motion variables as ${}^B\mathbf{v}^P = u_7\hat{\mathbf{a}}_1 + u_8\hat{\mathbf{a}}_2 + u_9\hat{\mathbf{a}}_3$ rather than as in Eq. (103).

The analyst may be interested in bringing some noncontributing forces into evidence and leaving others out of the picture. A selective exposition of such forces is accomplished by the choice of constraint equations that are dealt with. For example, if one is interested only in the component of $\mathbf{C}$ in the direction of $\hat{\mathbf{b}}_1$ then only the first of Eqs. (77) is formed and inspected to determine that the constraint force $\mathbf{C}_1 = \lambda_1\hat{\mathbf{b}}_1$ is applied to P, and that $-\mathbf{C}_1$ is applied to B at $\overline{B}$. This force is then brought into evidence by working only with the first of Eqs. (104) and not the second or third. The other two components of $\mathbf{C}$ are kept out of the light by leaving the second and third of Eqs. (77) unstated and by omitting introduction of additional motion variables u_8 and u_9.

CHAPTER 3

APPLICATIONS

A concise method is presented in Chapter 2 for inspecting constraint equations written at the velocity level to identify dot products of certain vectors and thereby determine the direction of a constraint force or torque, together with the point or body to which it must be applied. This one procedure, rigorously justified in Chapter 2, is uniformly applicable to all configuration constraints, including those involving prescribed positions, and also to motion constraints described by nonholonomic equations that are linear in the motion variables. What follows are illustrations of application of the method to various constraints commonly encountered in practice. Section 3.1 takes up the configuration constraints imposed on members of a multibody system by ideal joints, otherwise known as kinematic pairs. Two rigid bodies that roll without slipping on one another are subject to a motion constraint that is addressed in Sec. 3.2. Section 3.3 contains treatment of a simple example of prescribed motion. A variety of configuration and motion constraints are imposed simultaneously on the system dealt with in Sec. 3.4. In the example presented in Sec. 3.5, the multipliers introduced in the present method are seen to be precisely the measure numbers of constraint forces and torques brought into evidence by the introduction of certain additional motion variables. Finally, Kepler's first and second laws are treated as constraints in Sec. 3.6; it is shown that the resultant constraint force is in conformity with the universal law of gravitation.

3.1 Joints

The basic ideal joints used in models of multibody systems include revolute or pin joints, universal or Hooke's joints, spherical or ball-and-socket joints, prismatic or slider joints, planar joints, cylindrical joints, and so forth. Joints are also referred to in the literature as kinematic pairs; several are illustrated in Figure 5. The essence of a joint may be regarded in terms of figure and ground, a concept discussed in Ref. [35] in connection with music, drawings, and mathematics. One may think of the motion permitted by a joint as a figure, and of the motion prevented as the complementary ground, or vice versa. The total picture made up by the figure and ground in this case accounts for six degrees of freedom enjoyed by one body relative to another in the absence of a joint connecting the two.

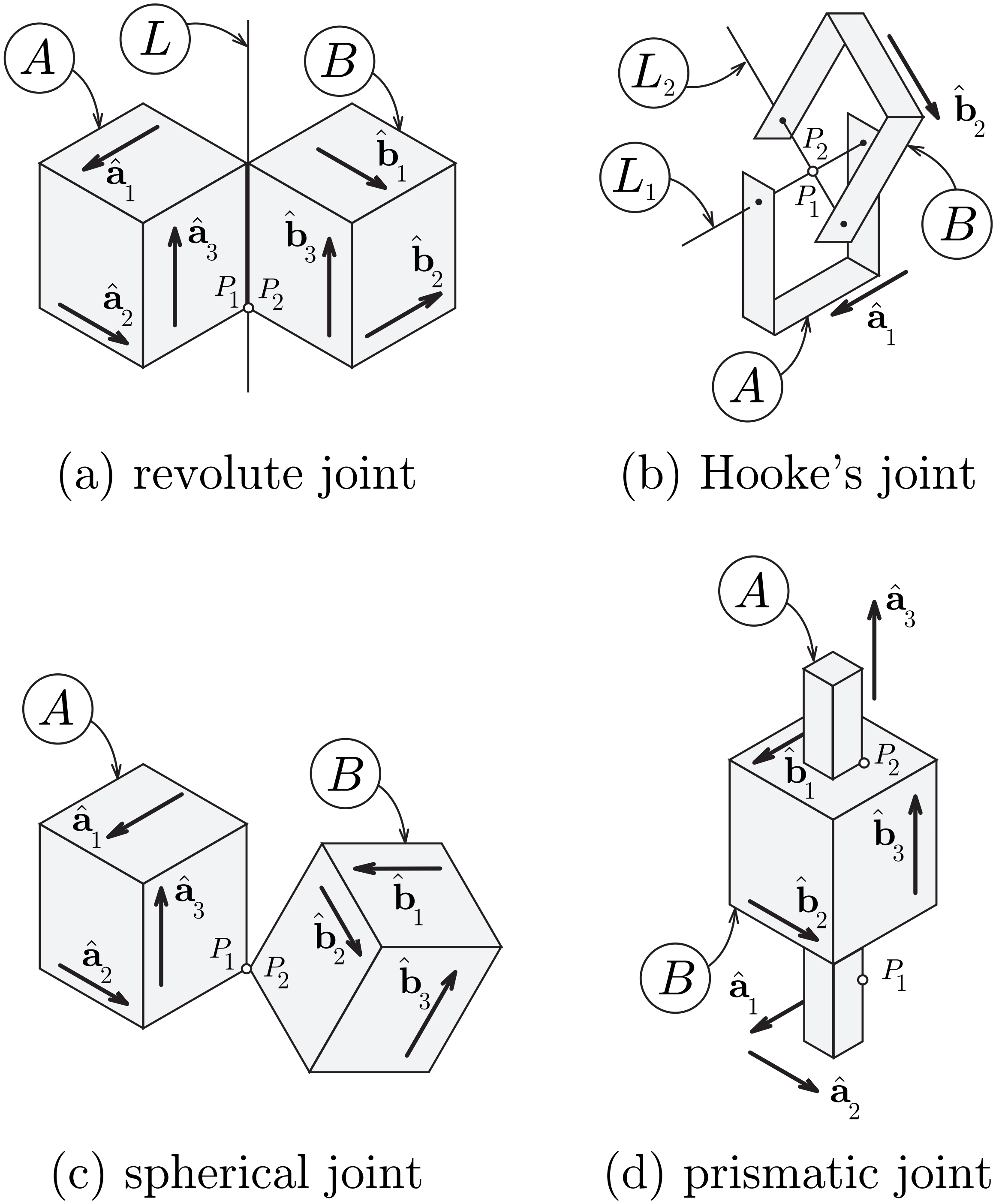
A joint that connects two rigid bodies A and B imposes a configuration constraint expressed with equations that belong to one of two classes; a restriction is placed either on relative orientation or on relative translation. The restrictions in the first class require orthogonality of two vectors $\mathbf{a}$ and $\mathbf{b}$ fixed respectively in A and B; the constraint may be expressed by up to three equations of the form

$$\mathbf{b} \cdot \mathbf{a} = 0 \tag{107}$$

The second class involves the relative translation of two points P_1 and P_2 that belong respectively to A and B; there may be up to three equations expressed as

$$(\mathbf{p}_2 - \mathbf{p}_1) \cdot \mathbf{a} = 0 \tag{108}$$

where $\mathbf{p}_1$ and $\mathbf{p}_2$ are the respective position vectors to P_1 and P_2 from a point O fixed in a Newtonian reference frame N, and where $\mathbf{a}$ is a vector fixed in A. Alternatively, a vector $\mathbf{b}$ fixed in B could serve in place of $\mathbf{a}$. Constraints imposed by the joints depicted in parts a, b, c, and d of Figure 5 are described by, respectively, 2, 1, 0, and 3 relationships having the form of Eqs. (107), and by 3, 3, 3, and 2 equations



(a) revolute joint

(b) Hooke's joint

(c) spherical joint

(d) prismatic joint

Figure 5: Examples of Joints

with the form of (108). As demonstrated presently, the relationships in both classes have the form of Eqs. (37); when they are differentiated with respect to time in N and inspected according to the procedure described in Sec. 2.4, the directions of the associated constraint forces and torques are obtained and the multipliers are unambiguously related to the constraint forces and torques.

Because vector $\mathbf{a}$ is fixed in A, one may always choose two particles P_3 and P_4 belonging to A such that $\mathbf{p}_4 = \mathbf{p}_3 + \mathbf{a}$. Likewise, one may always choose particles P_5 and P_6 belonging to B so that $\mathbf{p}_6 = \mathbf{p}_5 + \mathbf{b}$. Hence, Eq. (107) describing an orientation constraint can be seen to have the form of Eqs. (37),

$$g(\mathbf{p}_1, \ldots, \mathbf{p}_\nu, t) = (\mathbf{p}_6 - \mathbf{p}_5) \cdot (\mathbf{p}_4 - \mathbf{p}_3) = 0 \tag{109}$$

where $\mathbf{p}_i$ are the respective position vectors to the four particles P_i $(i = 3, 4, 5, 6)$ from a point O fixed in N. This holonomic constraint equation can be differentiated formally with the machinery of Eqs. (38) after constructing the partial derivatives

$$\begin{aligned}
\partial g / \partial \mathbf{p}_3 &= -(\mathbf{p}_6 - \mathbf{p}_5) \cdot \underline{\mathbf{U}} = -(\mathbf{p}_6 - \mathbf{p}_5) \\
\partial g / \partial \mathbf{p}_4 &= (\mathbf{p}_6 - \mathbf{p}_5) \cdot \underline{\mathbf{U}} = (\mathbf{p}_6 - \mathbf{p}_5) \\
\partial g / \partial \mathbf{p}_5 &= -\underline{\mathbf{U}} \cdot (\mathbf{p}_4 - \mathbf{p}_3) = -(\mathbf{p}_4 - \mathbf{p}_3) \\
\partial g / \partial \mathbf{p}_6 &= \underline{\mathbf{U}} \cdot (\mathbf{p}_4 - \mathbf{p}_3) = (\mathbf{p}_4 - \mathbf{p}_3) \\
\partial g / \partial t &= 0
\end{aligned} \tag{110}$$

where $\underline{\mathbf{U}}$ is the unit dyadic, and then assembling them according to Eqs. (39) as

$$\begin{aligned}
&{}^N\mathbf{v}^{P_3} \cdot (\mathbf{p}_5 - \mathbf{p}_6) + {}^N\mathbf{v}^{P_4} \cdot (\mathbf{p}_6 - \mathbf{p}_5) + {}^N\mathbf{v}^{P_5} \cdot (\mathbf{p}_3 - \mathbf{p}_4) + {}^N\mathbf{v}^{P_6} \cdot (\mathbf{p}_4 - \mathbf{p}_3) \\
&= ({}^N\mathbf{v}^{P_4} - {}^N\mathbf{v}^{P_3}) \cdot \mathbf{b} + ({}^N\mathbf{v}^{P_6} - {}^N\mathbf{v}^{P_5}) \cdot \mathbf{a} \\
&= ({}^N\boldsymbol{\omega}^A \times \mathbf{a}) \cdot \mathbf{b} + ({}^N\boldsymbol{\omega}^B \times \mathbf{b}) \cdot \mathbf{a} \\
&= ({}^N\boldsymbol{\omega}^B - {}^N\boldsymbol{\omega}^A) \cdot (\mathbf{b} \times \mathbf{a}) = 0
\end{aligned} \tag{111}$$

This result can be obtained more easily by a straightforward differentiation of Eq. (107) with respect to time in N; there is no need to take the route of Eqs. (38), whose main purpose is to establish the functional character of $\mathbf{W}_{is}$ and Y_s. As indicated in Sec. 2.4, inspection of this constraint equation written at the velocity level indicates that A exerts upon B a constraint couple whose torque is given by

$$\mathbf{T} = \lambda(\mathbf{b} \times \mathbf{a}) \tag{112}$$

and that B exerts upon A a constraint couple whose torque is given by $-\mathbf{T}$.

Differentiation of Eq. (108) with respect to time in N yields

$$\begin{aligned}({}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1}) \cdot \mathbf{a} + (\mathbf{p}_2 - \mathbf{p}_1) \cdot ({}^N\boldsymbol{\omega}^A \times \mathbf{a}) &= \\ ({}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1}) \cdot \mathbf{a} - {}^N\boldsymbol{\omega}^A \cdot [(\mathbf{p}_2 - \mathbf{p}_1) \times \mathbf{a}] &= 0\end{aligned} \tag{113}$$

Inspection of this constraint equation written at the velocity level shows that a constraint force given by

$$\mathbf{C} = \lambda\mathbf{a} \tag{114}$$

is applied to P_2, whereas a constraint force $-\mathbf{C}$ is applied to P_1. Furthermore, A is acted upon by a constraint couple whose torque is given by

$$\mathbf{T} = -\lambda(\mathbf{p}_2 - \mathbf{p}_1) \times \mathbf{a} = -(\mathbf{p}_2 - \mathbf{p}_1) \times \mathbf{C} \tag{115}$$

The foregoing results for restrictions on orientation and translation dictated by joints can now be applied to specific cases.

3.1.1 Revolute Joint

A revolute joint, also known as a hinge joint or a pin joint, permits rotation of B relative to A about a line L that is fixed in both A and B. Without the joint, the movement of one body relative to the other would be completely unrestricted and B would possess six degrees of freedom in A. The joint permits one degree of freedom of B relative to A, therefore the number M of holonomic constraint equations must be 5. Relative translation of points on L is prevented; if points P_1 of A and P_2 of

B lie on L as shown in Figure 5a, they must remain coincident. In addition, certain changes in relative orientation are prevented.

The procedure given in Sec. 2.4 can be used to develop expressions in vector form for a single force and the torque of a couple that, together, are equivalent to the set of forces exerted by A on B. At the same time the procedure produces expressions for a force and torque exerted by B on A.

The requirement that P_1 be coincident with P_2 can be expressed with a vector equation at the position level as $\mathbf{p}_2 - \mathbf{p}_1 = \mathbf{0}$ or at the velocity level as ${}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1} = \mathbf{0}$. Three scalar equations can be constructed from the latter vector equation,

$$({}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1}) \cdot \hat{\mathbf{a}}_s = 0 \qquad (s = 1, 2, 3) \tag{116}$$

where $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, and $\hat{\mathbf{a}}_3$ can be a set of nonparallel, noncoplanar (but not necessarily mutually perpendicular) unit vectors fixed in any reference frame whatsoever. By letting $\hat{\mathbf{a}}_s$ play the part of $\mathbf{a}$ in Eq. (114) one at a time, and by realizing that the vector $\mathbf{p}_2 - \mathbf{p}_1$ vanishes in Eq. (115), it can be concluded that in order to bring about coincidence of P_1 and P_2, a constraint force applied to B at P_2 is given by

$$\mathbf{C} = \lambda_1 \hat{\mathbf{a}}_1 + \lambda_2 \hat{\mathbf{a}}_2 + \lambda_3 \hat{\mathbf{a}}_3 \tag{117}$$

and $-\mathbf{C}$ is applied to A at P_1. No constraint torque is required to enforce coincidence.

Henceforth, let $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, and $\hat{\mathbf{a}}_3$ be a set of right-handed, mutually perpendicular unit vectors fixed in A, with $\hat{\mathbf{a}}_3$ parallel to L. Likewise, let $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, and $\hat{\mathbf{b}}_3$ be a similar set of unit vectors fixed in B such that $\hat{\mathbf{b}}_3 = \hat{\mathbf{a}}_3$. Two restrictions on orientation imposed by the joint can be expressed as

$$\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{a}}_{s-3} = 0 \qquad (s = 4, 5) \tag{118}$$

In view of Eq. (112), with the roles of $\mathbf{a}$ and $\mathbf{b}$ first played by unit vectors $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{b}}_3$ respectively, and then by $\hat{\mathbf{a}}_2$ and $\hat{\mathbf{b}}_3$, a constraint torque given by

$$\mathbf{T} = \lambda_4 \hat{\mathbf{b}}_3 \times \hat{\mathbf{a}}_1 + \lambda_5 \hat{\mathbf{b}}_3 \times \hat{\mathbf{a}}_2 \tag{119}$$

is applied to B. When the constraints are satisfied, $\hat{\mathbf{b}}_3 \times \hat{\mathbf{a}}_1 = \hat{\mathbf{a}}_2$ and $\hat{\mathbf{b}}_3 \times \hat{\mathbf{a}}_2 = -\hat{\mathbf{a}}_1$, therefore

$$\mathbf{T} = \lambda_4 \hat{\mathbf{a}}_2 - \lambda_5 \hat{\mathbf{a}}_1 \tag{120}$$

A constraint torque $-\mathbf{T}$ is exerted upon A. One typically assumes that the bearing surfaces of an ideal revolute joint are perfectly smooth, in which case $\mathbf{T}$ should have no component in the direction of $\hat{\mathbf{a}}_3$; the expression obtained for $\mathbf{T}$ meets this expectation.

3.1.2 Hooke's Joint

Hooke's joint shown in Figure 5b, also called a universal joint or a Cardan joint, permits a rigid body B to perform two successive simple rotations relative to rigid body A, first about a line L_1 fixed in A, and then about a line L_2 fixed in B. Lines L_1 and L_2 are perpendicular, fixed in a rigid cross, and intersect at coincident points P_1 of A and P_2 of B. The universal joint permits B two degrees of freedom in A, thus the configuration constraint is described by four ($M = 4$) holonomic constraint equations.

The requirement that P_1 and P_2 remain coincident has already been considered in connection with the revolute joint; consequently, Eqs. (116) remain applicable and the constraint force $\mathbf{C}$ applied by A to B at P_2 is given by Eq. (117) as before.

The two restrictions on relative orientation dictated by the revolute joint give way to a single restriction in the case of Hooke's joint. Lines L_1 and L_2 must remain perpendicular, and this configuration constraint can be expressed as

$$\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{a}}_1 = 0 \tag{121}$$

where unit vector $\hat{\mathbf{a}}_1$ is now assumed to be parallel to L_1, and $\hat{\mathbf{b}}_2$ is a unit vector fixed in B and parallel to L_2. In view of Eq. (112), A exerts upon B a constraint couple whose torque is given by

$$\mathbf{T} = \lambda_4 (\hat{\mathbf{b}}_2 \times \hat{\mathbf{a}}_1) \tag{122}$$

and B exerts upon A a constraint couple whose torque is given by $-\mathbf{T}$. In this way, $\mathbf{T}$ is shown to be perpendicular both to $\hat{\mathbf{a}}_1$ and to $\hat{\mathbf{b}}_2$, in agreement with Eq. (4.3.11) of Ref. [44] and the description on p. 161 of Ref. [31]. The present method enables one to arrive at this conclusion without having to construct free-body diagrams of A, B, and the rigid cross, consider explicitly the smoothness and the orientations of the bearings in A and B that support the cross, and then take the law of action and reaction into account.

3.1.3 Spherical Joint

The spherical joint depicted in Figure 5c, otherwise known as a ball-and-socket joint, permits B to have three degrees of freedom relative to A and thus imposes a configuration constraint described by three ($M = 3$) holonomic constraint equations. Although it does not place any restrictions on the relative orientation of the two bodies, the joint confines point P_2 of B to be coincident with point P_1 of A. Once again, Eqs. (116) remain applicable and the constraint force $\mathbf{C}$ applied by A to B at P_2 is given by Eq. (117).

3.1.4 Prismatic Joint

The purpose of a prismatic joint, also called a translational joint or slider joint, is to permit translation of B relative to A in one direction as shown in Figure 5d. The joint inhibits any change in relative orientation, as well as translation in any direction orthogonal to the axis of the joint. One degree of freedom of B relative to A is allowed, therefore the number M of holonomic constraint equations must be 5.

Because the orientation of B relative to A cannot change, the relative angular velocity must vanish. This condition can be expressed with the vector equation ${}^A\boldsymbol{\omega}^B = \mathbf{0}$, alternatively written as ${}^N\boldsymbol{\omega}^B - {}^N\boldsymbol{\omega}^A = \mathbf{0}$. Three scalar equations can be constructed from this vector equation,

$$({}^N\boldsymbol{\omega}^B - {}^N\boldsymbol{\omega}^A) \cdot \hat{\mathbf{a}}_s = 0 \qquad (s = 1, 2, 3) \tag{123}$$

where $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, and $\hat{\mathbf{a}}_3$ can be a set of nonparallel, noncoplanar (but not necessarily mutually perpendicular) unit vectors fixed in any reference frame whatsoever. According to the instructions given in Sec. 2.4, inspection of these constraint equations expressed at the velocity level leads immediately to an expression for $\mathbf{T}$,

$$\mathbf{T} = \lambda_1 \hat{\mathbf{a}}_1 + \lambda_2 \hat{\mathbf{a}}_2 + \lambda_3 \hat{\mathbf{a}}_3 \tag{124}$$

the constraint torque exerted by A on B in order to prevent a change in relative orientation.

Henceforth, let $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, and $\hat{\mathbf{a}}_3$ be a set of right-handed, mutually perpendicular unit vectors fixed in A, with $\hat{\mathbf{a}}_3$ parallel to the axis of the joint. Likewise, let $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, and $\hat{\mathbf{b}}_3$ be a similar set of unit vectors fixed in B such that $\hat{\mathbf{b}}_3 = \hat{\mathbf{a}}_3$. Two restrictions on translation imposed by the joint can be expressed as

$$(\mathbf{p}_2 - \mathbf{p}_1) \cdot \hat{\mathbf{a}}_{s-3} = 0 \qquad (s = 4, 5) \tag{125}$$

With the role of $\mathbf{a}$ in Eqs. (108) and (114) played first by $\hat{\mathbf{a}}_1$ and then by $\hat{\mathbf{a}}_2$, the constraint force $\mathbf{C}$ applied by A on B at P_2 is given by

$$\mathbf{C} = \lambda_4 \hat{\mathbf{a}}_1 + \lambda_5 \hat{\mathbf{a}}_2 \tag{126}$$

The surfaces of an ideal prismatic joint are assumed to be perfectly smooth and therefore $\mathbf{C}$ should have no component in the direction of $\hat{\mathbf{a}}_3$. The expression obtained for $\mathbf{C}$ is observed to agree with such a model. A constraint force $-\mathbf{C}$ is applied by B to A at P_1; in addition, Eqs. (115) indicate that A is subject to a constraint couple whose torque is given by $(\mathbf{p}_2 - \mathbf{p}_1) \times (-\mathbf{C})$. Now, the set of forces consisting of $-\mathbf{C}$ bound to P_1, and the constraint couple, are seen to be equivalent to a single force $-\mathbf{C}$ applied to $\bar{A}$, the point of A that is instantaneously coincident with P_2. In fact, equivalence of the two sets of forces corresponds to the existence of an equivalent way of expressing the restrictions on translation at the velocity level: two relationships obtained from Eq. (113), with the role of $\mathbf{a}$ played first by $\hat{\mathbf{a}}_1$ and then by $\hat{\mathbf{a}}_2$, are equivalent to

$$({}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{\bar{A}}) \cdot \hat{\mathbf{a}}_{s-3} = 0 \qquad (s = 4, 5) \tag{127}$$

Inspection of these two equations according to the method explained in Sec. 2.4 indicates $\mathbf{C} = \lambda_4\hat{\mathbf{a}}_1 + \lambda_5\hat{\mathbf{a}}_2$ is applied to B at P_2, and $-\mathbf{C}$ is applied to A at $\bar{A}$.

3.2 Rolling

If points P_1 and P_2 are in contact with each other, and belong respectively to rigid bodies A and B, and if the velocities of P_1 and P_2 in any reference frame are equal to each other at a certain instant of time, then A and B are said to be rolling on each other at that instant. The absence of a difference in velocities means no slipping is taking place. Although velocities in any reference frame can be equated, application of the material contained in Sec. 2.4 is made possible by use of a Newtonian reference frame N to express the condition of rolling, ${}^N\mathbf{v}^{P_2} = {}^N\mathbf{v}^{P_1}$ or ${}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1} = \mathbf{0}$. Three scalar constraint equations can be obtained from the vector equation by forming scalar products with dextral, mutually perpendicular unit vectors $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, and $\hat{\mathbf{a}}_3$

$$({}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1}) \cdot \hat{\mathbf{a}}_s = 0 \qquad (s = 1, 2, 3) \tag{128}$$

There exists a plane that is tangent to the surface of A at P_1, and to the surface of B at P_2; unit vector $\hat{\mathbf{a}}_1$ is normal to this plane, whereas $\hat{\mathbf{a}}_2$ and $\hat{\mathbf{a}}_3$ are parallel to it. The first of Eqs. (128) is a holonomic constraint equation expressed at the velocity level because it describes a restriction on the positions of P_1 and P_2; the former may not penetrate B and the latter may not penetrate A. The second and third of Eqs. (128) are nonholonomic constraint equations. Nevertheless, because all of the constraint equations are expressed at the velocity level they can be inspected according to the directions given in Sec. 2.4 to obtain the constraint force

$$\mathbf{C} = \lambda_1\hat{\mathbf{a}}_1 + \lambda_2\hat{\mathbf{a}}_2 + \lambda_3\hat{\mathbf{a}}_3 \tag{129}$$

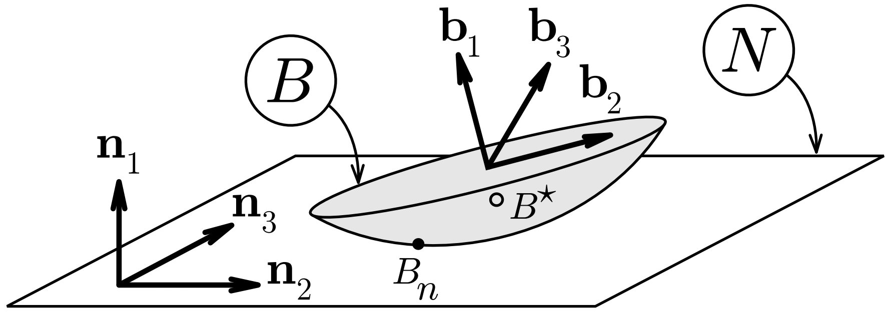


Figure 6: Rattleback

applied by A to B at P_2. In turn, a constraint force $-\mathbf{C}$ is exerted by B upon A at P_1. The components $\lambda_1\hat{\mathbf{a}}_1$ and $\lambda_2\hat{\mathbf{a}}_2 + \lambda_3\hat{\mathbf{a}}_3$ of $\mathbf{C}$ are, respectively, typically referred to as the normal force and the tangential or frictional force.

A rattleback or wobblestone is an object with an ellipsoidal surface, as shown in Figure 6. When placed on a rough horizontal surface and rotated about a vertical axis, it can stop rotating, begin to shudder violently, and then start rotating in the opposite direction. Mitiguy and Banerjee present an analysis of the motion of a rattleback in Ref. [53]. Although their work does not propose a strong connection between the forms of constraint equations and the contact force acting on the rattleback, such an association is established with the foregoing results. In their study N plays the part of A, and P_1 is the point of N that is instantaneously in contact with the rattleback B at B_n, the point that plays the part of P_2. Constraint equations (25)–(27) in Ref. [53] are written as $\mathbf{v}^{B_n} \cdot \mathbf{n}_s = 0$, where $\mathbf{v}^{B_n}$ denotes the velocity of B_n in N, and $\mathbf{n}_s$ $(s = 1, 2, 3)$ are dextral, mutually perpendicular unit vectors fixed in N. Those equations make implicit use of the fact that ${}^N\mathbf{v}^{P_1} = \mathbf{0}$ and thus have the form of Eqs. (128) here. The expression for the contact force given in Eq. (22) of Ref. [53] is a counterpart to what appears in Eq. (129), where the respective symbols $\mathbf{F}^{B_n}$, τ_s, and $\mathbf{n}_s$ are used in place of $\mathbf{C}$, λ_s, and $\hat{\mathbf{a}}_s$. It can be verified that the six elements of the matrix product $-XF$, where X and F are reported in Eqs. (31) and (29) of Ref. [53], are identical to $(F_r)_{\mathcal{C}}$ $(r = 1, \ldots, 6)$ obtained from Eqs. (72) here.

3.3 Prescribed Motion

A configuration constraint may be imposed on a particle P by requiring it to move in a reference frame A such that the position vector from a point fixed in A to P is a known function of the time t. In this case the velocity ${}^{A}\mathbf{v}^{P}$ of P in A is also a known function of t, as is the acceleration ${}^{A}\mathbf{a}^{P}$ of P in A, and P is said to undergo prescribed motion in A.

The constraint force needed to move a particle according to a prescribed schedule can be identified through the process of inspection described in Sec. 2.2.1. A simple example is provided by a particle P whose motion in a Newtonian reference frame N is considered to be unconstrained when no contact or distance forces act on P. Let $\hat{\mathbf{n}}$ be a unit vector fixed in N, and suppose that P is required to move such that the position vector $\mathbf{p}$ from a point fixed in N to P satisfies the relationship

$$\mathbf{p} \cdot \hat{\mathbf{n}} = -\frac{1}{2}gt^2 + v_0 t + p_0 \tag{130}$$

where g is a constant with units of force per unit mass, and where the constants p_0 and v_0 are, respectively, the values of $\mathbf{p} \cdot \hat{\mathbf{n}}$ and ${}^{N}\mathbf{v}^{P} \cdot \hat{\mathbf{n}}$ at $t = 0$. This rheonomic holonomic constraint equation can be brought into the form of Eqs. (39) by differentiation with respect to t in N,

$${}^{N}\mathbf{v}^{P} \cdot \hat{\mathbf{n}} + gt - v_0 = 0 \tag{131}$$

Inspection of this constraint equation written at the velocity level indicates that a constraint force

$$\mathbf{C} = \lambda \hat{\mathbf{n}} \tag{132}$$

must be applied to P. Differentiation of Eq. (131) with respect to t in N yields an equation

$${}^{N}\mathbf{a}^{P} \cdot \hat{\mathbf{n}} + g = 0 \tag{133}$$

that, upon comparison with Eqs. (41), allows one to identify $Z = g$. In the absence of all other contact and distance forces (that is, $\mathbf{f} = \mathbf{0}$), the value of λ is found with the aid of Eq. (11) to be

$$\lambda = -\frac{g + (\mathbf{0} \cdot \hat{\mathbf{n}})/m}{\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}/m} = -mg \tag{134}$$

where m is the mass of P.

3.4 Several Constraints

A system may be subject to several types of constraints at once, and they can all be dealt with in a uniform manner by applying the procedure given in Sec. 2.4. The example presented in Sec. 2.13 of Ref. [44] involves a rigid body consisting of a small block and a small sharp-edged circular disk connected by a rigid rod, as shown in Figure 7; the body is confined to remain between two parallel panes of glass that are attached to each other and made to rotate at a prescribed angular speed. This example is used to demonstrate the process of inspecting a variety of constraint equations to obtain information about forces needed to impose a motion constraint and configuration constraints that include the condition of prescribed motion.

The block and the disk are treated as particles P_1 and P_2. A set of dextral, mutually perpendicular unit vectors $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$ are fixed in the rigid body formed by P_1, P_2, and the rod R, with $\hat{\mathbf{e}}_x$ having the direction of the position vector from P_1 to P_2, and $\hat{\mathbf{e}}_z$ normal to the panes of glass B as shown in Figure 7. The body B is made to rotate at a constant rate ω about a line Y fixed both in B and in a Newtonian reference frame N.

First, consider the two configuration constraints in which the position vectors $\mathbf{p}_s$ from a point O fixed in N to P_s $(s = 1, 2)$ are each prevented by the panes of glass from having a component in the direction of $\hat{\mathbf{e}}_z$. The conditions are expressed with the two relationships

$$\mathbf{p}_s \cdot \hat{\mathbf{e}}_z = 0 \qquad (s = 1, 2) \tag{135}$$

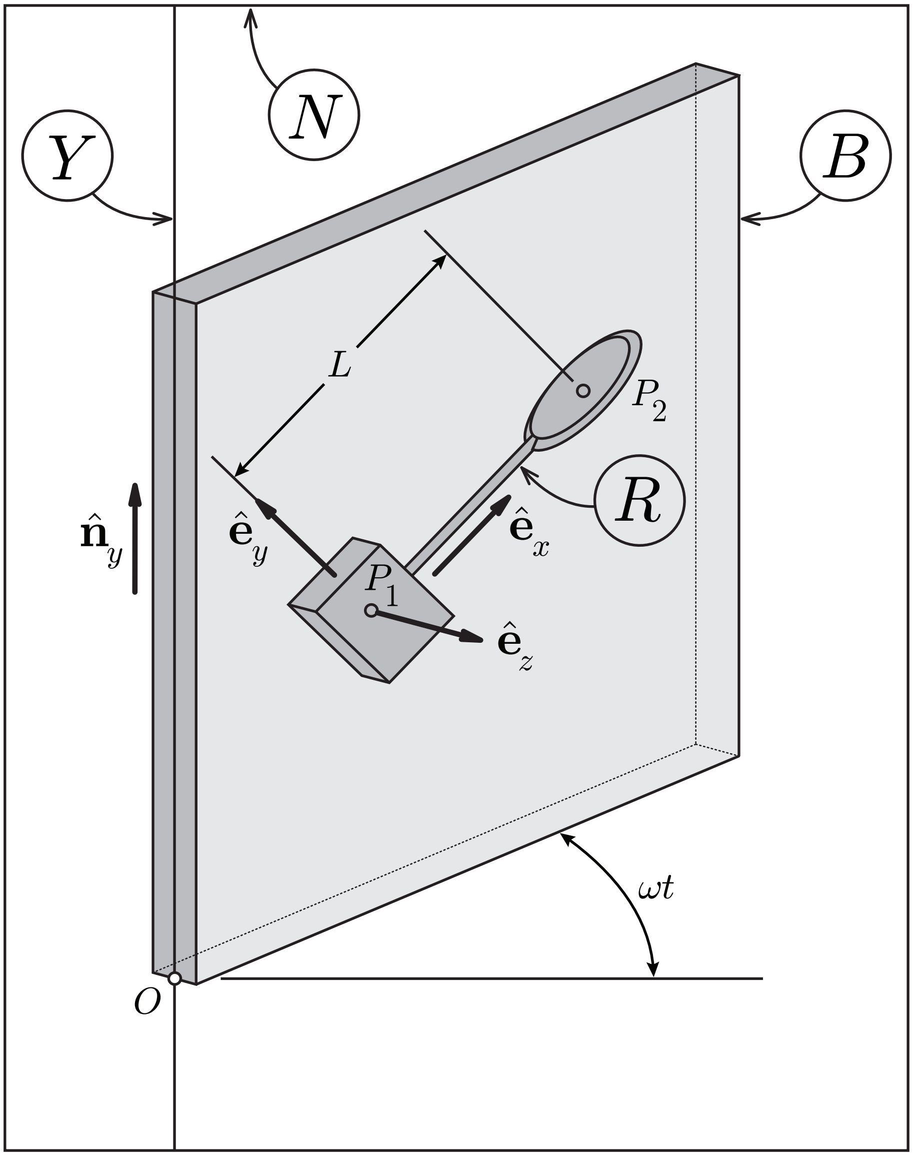


Figure 7: Two Particles Confined in Parallel Panes of Glass

Differentiation with respect to time in N yields, after recognizing that unit vector $\hat{\mathbf{e}}_z$ is fixed in B, ${}^N\mathbf{v}^{P_s} \cdot \hat{\mathbf{e}}_z + \mathbf{p}_s \cdot \omega \hat{\mathbf{n}}_y \times \hat{\mathbf{e}}_z = 0$, where unit vector $\hat{\mathbf{n}}_y$ is parallel to the line Y. Now, the second term can be rewritten $\mathbf{p}_s \cdot \omega \hat{\mathbf{n}}_y \times \hat{\mathbf{e}}_z = -\omega \hat{\mathbf{n}}_y \times \mathbf{p}_s \cdot \hat{\mathbf{e}}_z = -{}^N\mathbf{v}^{\overline{B}_s} \cdot \hat{\mathbf{e}}_z$, where $\overline{B}_s$ is the point of B that coincides with P_s. Hence, two velocity level constraint equations are expressed as

$$({}^N\mathbf{v}^{P_s} - {}^N\mathbf{v}^{\overline{B}_s}) \cdot \hat{\mathbf{e}}_z = 0 \qquad (s = 1, 2) \tag{136}$$

Inspection signifies a constraint force

$$\mathbf{C}_{ss} = \lambda_s \hat{\mathbf{e}}_z \qquad (s = 1, 2) \tag{137}$$

is applied to particle P_s, and $-\mathbf{C}_{ss}$ is applied to $\overline{B_s}$.

A third configuration constraint is present: P_1 and P_2 are required to belong to a rigid body and maintain a constant distance L from one another. As demonstrated in Sec. 2.2.1, this restriction leads to the velocity level constraint equation

$$({}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1}) \cdot L\hat{\mathbf{e}}_x = 0 \tag{138}$$

Inspection of this relationship indicates a constraint force

$$\mathbf{C}_{23} = \lambda_3 L \hat{\mathbf{e}}_x \tag{139}$$

is applied to P_2, and $\mathbf{C}_{13} = -\mathbf{C}_{23}$ is applied to P_1.

The fourth constraint is a motion constraint and hence immediately written at the velocity level. The sharp edge of the disk prevents the velocity ${}^B\mathbf{v}^{P_2}$ of P_2 in B from having a component in the direction of $\hat{\mathbf{e}}_y$. However, ${}^B\mathbf{v}^{P_2}$ can be written as ${}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{\overline{B}_2}$; therefore the constraint equation is written as

$$({}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{\overline{B}_2}) \cdot \hat{\mathbf{e}}_y = 0 \tag{140}$$

and inspection reveals a constraint force

$$\mathbf{C}_{24} = \lambda_4 \hat{\mathbf{e}}_y \tag{141}$$

is applied to P_2 whereas $-\mathbf{C}_{24}$ is applied to $\overline{B_2}$.

The resultant constraint forces acting on P_1 and P_2 are given by

$$\mathbf{C}_1 = \lambda_1 \hat{\mathbf{e}}_z - \lambda_3 L \hat{\mathbf{e}}_x, \qquad \mathbf{C}_2 = \lambda_2 \hat{\mathbf{e}}_z + \lambda_3 L \hat{\mathbf{e}}_x + \lambda_4 \hat{\mathbf{e}}_y \tag{142}$$

3.5 Constraint Force Measure Numbers

The procedure for bringing noncontributing forces into evidence is explained and illustrated by example in Sec. 4.9 of Ref. [44]. Seven additional motion variables are introduced to unveil seven measure numbers of constraint forces and torques acting on a system with two degrees of freedom. It is reassuring to revisit this example and see that the process of inspection presented in Sec. 2.4 gives rise to multipliers that are in fact the measure numbers brought into equations of motion by Kane's procedure.

A particle P slides freely on a smooth rod R that is attached at an angle β to a sleeve S, as depicted in Figure 8. The sleeve is supported by a smooth vertical shaft V and a smooth bearing surface B, both of which are fixed in a Newtonian reference frame N. Two dextral, mutually perpendicular sets of unit vectors are defined to facilitate analysis. Unit vector $\hat{\mathbf{s}}_1$ is directed vertically downward, $\hat{\mathbf{s}}_2$ is parallel to the plane determined by the axes of S and R, and $\hat{\mathbf{s}}_3 = \hat{\mathbf{s}}_1 \times \hat{\mathbf{s}}_2$ completes the first set. The members of the second set include $\hat{\mathbf{r}}_1$ parallel to the axis of R, $\hat{\mathbf{r}}_2$ parallel to the plane containing the axes of S and R, and $\hat{\mathbf{r}}_3 = \hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2$.

The system composed of P, R, and S possesses two degrees of freedom in N; the rigid body C formed by S and R is permitted to have an angular velocity ${}^N\boldsymbol{\omega}^C$ in N parallel to $\hat{\mathbf{s}}_1$, and P is permitted to have a velocity ${}^R\mathbf{v}^P$ in R parallel to $\hat{\mathbf{r}}_1$. A complementary way to describe the system, which contains a revolute joint and a prismatic joint, is to note that ${}^N\boldsymbol{\omega}^C$ is prevented from having a component perpendicular to $\hat{\mathbf{s}}_1$, ${}^R\mathbf{v}^P$ is prevented from having a component perpendicular to $\hat{\mathbf{r}}_1$, and a point O fixed in C at the intersection of the axes of S and R is prevented

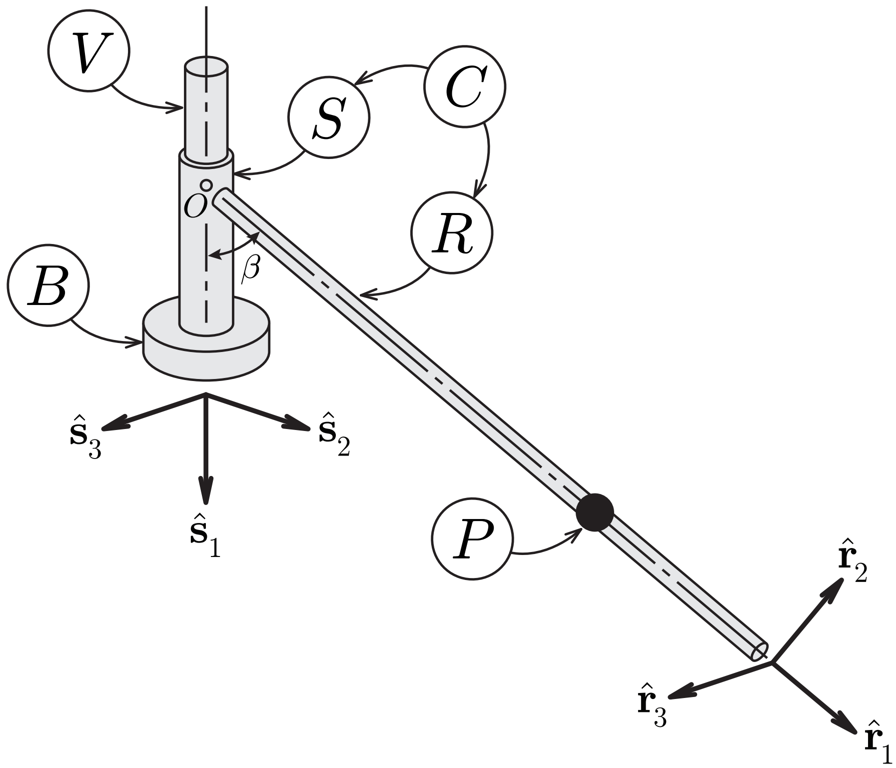


Figure 8: Particle Sliding on a Smooth Rod

from having any velocity in N. Such a description is easily given at the velocity level even though the system is subject to configuration constraints rather than motion constraints.

The process of introducing additional motion variables is inherently one of expressing constraint equations at the velocity level. Motion that the system can undergo is characterized by two motion variables u_1 and u_2. In the course of the example additional motion variables that are in fact zero are introduced as $u_3, \ldots, u_9$. The additional motion variables u_3, u_4, and u_5 are introduced first as a group; they can be defined with constraint equations at the velocity level in terms of dot products by referring respectively to Eqs. (4.9.11), (4.9.12), and (4.9.13) together with (4.9.15).

$$u_3 = {}^N\mathbf{v}^O \cdot \hat{\mathbf{s}}_1 = 0 \tag{143}$$

$$u_4 = {}^N\boldsymbol{\omega}^C \cdot \hat{\mathbf{s}}_2 = 0 \tag{144}$$

$$u_5 = {}^R\mathbf{v}^P \cdot \hat{\mathbf{r}}_3 = ({}^N\mathbf{v}^P - {}^N\mathbf{v}^{\overline{R}}) \cdot \hat{\mathbf{r}}_3 = 0 \tag{145}$$

where $\overline{R}$ is the point of R with which P is in contact. Inspection of Eq. (143) according to the instructions in Sec. 2.4 shows that a constraint force $\lambda_1\hat{\mathbf{s}}_1$ is applied to C at point O. In view of Eq. (144), C is subject to a constraint couple whose torque is given by $\lambda_2\hat{\mathbf{s}}_2$. Finally, inspection of Eqs. (145) indicates a constraint force $\lambda_3\hat{\mathbf{r}}_3$ is applied by R to P, and $-\lambda_3\hat{\mathbf{r}}_3$ is applied by P to R at $\overline{R}$.

The example concludes with the introduction of four more additional motion variables u_6, u_7, u_8, and u_9 that, with reference to Eqs. (4.9.22)–(4.9.24), are defined through the following constraint equations

$$u_{s+2} = {}^N\mathbf{v}^O \cdot \hat{\mathbf{s}}_{s-2} = 0 \qquad (s = 4, 5) \tag{146}$$

$$u_8 = {}^N\boldsymbol{\omega}^C \cdot \hat{\mathbf{s}}_3 = 0 \tag{147}$$

$$u_9 = {}^R\mathbf{v}^P \cdot \hat{\mathbf{r}}_2 = ({}^N\mathbf{v}^P - {}^N\mathbf{v}^{\overline{R}}) \cdot \hat{\mathbf{r}}_2 = 0 \tag{148}$$

These relationships lead, by way of the process of inspection, to the expressions $\lambda_4\hat{\mathbf{s}}_2 + \lambda_5\hat{\mathbf{s}}_3$ for a constraint force applied to C at O, to $\lambda_6\hat{\mathbf{s}}_3$ for the torque of a constraint couple applied to C, and finally to $\lambda_7\hat{\mathbf{r}}_2$ for a constraint force applied by R to P, in addition to a constraint force $-\lambda_7\hat{\mathbf{r}}_2$ applied by P to R at $\overline{R}$.

The resultant constraint force $\boldsymbol{\sigma}$ applied to C at O is given by

$$\boldsymbol{\sigma} = \lambda_1\hat{\mathbf{s}}_1 + \lambda_4\hat{\mathbf{s}}_2 + \lambda_5\hat{\mathbf{s}}_3 \tag{149}$$

This is seen to be in agreement with Eq. (4.9.9); the multipliers λ_1, λ_4, and λ_5 are simply the measure numbers σ_1, σ_2, and σ_3, respectively. The resultant torque $\boldsymbol{\tau}$ of a constraint couple acting on C is made up of two components

$$\boldsymbol{\tau} = \lambda_2\hat{\mathbf{s}}_2 + \lambda_6\hat{\mathbf{s}}_3 \tag{150}$$

where the multipliers λ_2 and λ_6 are identical to the measure numbers τ_2 and τ_3 appearing in Eq. (4.9.8). Finally, the resultant constraint force $\boldsymbol{\rho}$ applied by R to P is written as

$$\boldsymbol{\rho} = \lambda_7\hat{\mathbf{r}}_2 + \lambda_3\hat{\mathbf{r}}_3 \tag{151}$$

in concert with Eq. (4.9.10), where λ_7 and λ_3 play the parts of the measure numbers ρ_2 and ρ_3. A constraint force $-\boldsymbol{\rho}$ is applied by P to R at $\overline{R}$. As noted in Ref. [44], the presumption of smoothness of all contact surfaces leads one not to expect a component of $\boldsymbol{\tau}$ in the direction of $\hat{\mathbf{s}}_1$ or a component of $\boldsymbol{\rho}$ in the direction of $\hat{\mathbf{r}}_1$.

It is worth noting the additional motion variables $u_3, \ldots, u_9$ that are introduced are in fact zero for the actual motion of the system. As pointed out in Sec. 2.5, a consequence of this is that constraint force measure numbers only contribute to the generalized active forces associated with the additional motion variables, and only one measure number appears in each such generalized active force. Examination of Eqs. (4.9.6), (4.9.7), (4.9.19)–(4.9.21), and (4.9.25)–(4.9.28) confirms that this is indeed the case.

3.6 The Universal Law of Gravitation

One of Sir Isaac Newton's profound accomplishments in the *Philosophiae Naturalis Principia Mathematica* is his demonstration that two-body orbital motion is the result of his universal law of gravitation. According to that law, the force of mutual gravitational attraction of two particles for one another is proportional to the product of their masses and inversely proportional to the square of the distance between them. Two-body orbital motion involves two particles P_1 and P_2 (or two spheres with uniform mass distribution) and takes place according to Kepler's laws, which can be considered as constraints. Kepler's first law states that the orbit of a planet, P_2, is an ellipse with the Sun, P_1, at a focus. More generally, the orbit may be described by a conic section. The second law holds that the position vector from P_1 to P_2 sweeps out equal areas in equal increments of time. After expressing Kepler's first and second laws with constraint equations, the material in Sec. 2.2 can be used to obtain information about the forces needed to impose the two laws. Specifically, it can be shown that the second law limits the constraint force resultant to be parallel

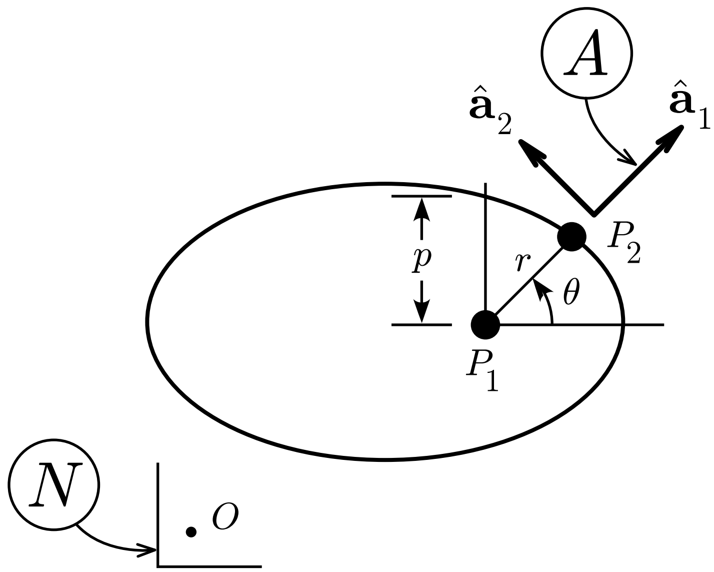


Figure 9: Two Particles Whose Relative Displacement is Described by an Ellipse

to the position vector from P_1 to P_2. It is then shown that the first law requires a constraint force of the form specified by Newton's law of universal gravitation.

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the respective position vectors to P_1 and P_2 from a point O fixed in a Newtonian reference frame N, and denote the distance between the two particles by r as illustrated in Fig. 9. The position vector from P_1 to P_2 can then be written as

$$\mathbf{p}_2 - \mathbf{p}_1 = r\hat{\mathbf{a}}_1 \tag{152}$$

where unit vector $\hat{\mathbf{a}}_1$ is fixed in a reference frame A whose angular velocity in N is ${}^N\boldsymbol{\omega}^A = \dot{\theta}\hat{\mathbf{a}}_3$. The motion of the two-body system takes place in a plane perpendicular to unit vector $\hat{\mathbf{a}}_3$, which is fixed in N. Differentiation of this expression with respect to time in N yields

$${}^N\mathbf{v}^{P_2/P_1} \triangleq {}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1} = \dot{r}\hat{\mathbf{a}}_1 + r\dot{\theta}\hat{\mathbf{a}}_2 \tag{153}$$

where $\hat{\mathbf{a}}_2 = \hat{\mathbf{a}}_3 \times \hat{\mathbf{a}}_1$. In standard texts of celestial mechanics and astrodynamics (for example, Ref. [12]) the quantity ${}^N\mathbf{v}^{P_2/P_1}$ is referred to as *the velocity*; it is important not to lose sight of the fact that it is in fact a difference in inertial velocities, and it may formally be referred to as the velocity of P_2 relative to P_1 in N, as in Ref. [44]. It is likewise essential to keep in mind that what is often called *the acceleration* is in fact a difference in inertial accelerations.

$$ {}^{N}\mathbf{a}^{P_2/P_1} \triangleq {}^{N}\mathbf{a}^{P_2} - {}^{N}\mathbf{a}^{P_1} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{a}}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{a}}_2 \tag{154} $$

The time rate of change of area swept out by $r\hat{\mathbf{a}}_1$ can be denoted by $\dot{A}$. According to Kepler's second law, it is a constant. The law can be expressed as (Ref. [12], p. 32, or Ref. [22], p. 213)

$$ \dot{A} = \frac{1}{2}r^2\dot{\theta} = \frac{r}{2}\,{}^{N}\mathbf{v}^{P_2/P_1} \cdot \hat{\mathbf{a}}_2 \tag{155} $$

or as a constraint equation at the velocity level,

$$ ({}^{N}\mathbf{v}^{P_2} - {}^{N}\mathbf{v}^{P_1}) \cdot \hat{\mathbf{a}}_2 - \frac{2\dot{A}}{r} = 0 \tag{156} $$

According to the results of Sec. 2.2, inspection of this equation reveals a constraint force

$$ \mathbf{C}_2 = \lambda_2\hat{\mathbf{a}}_2 \tag{157} $$

must be applied to P_2, and $-\mathbf{C}_2$ must be applied to P_1 if they are to obey Kepler's second law.

According to Kepler's first law, a time history of the position vector $r\hat{\mathbf{a}}_1$ can be described as an ellipse. Generally speaking it can be described by one of four types of conic sections, thus a holonomic constraint equation is given by [Ref. [12], Eq. (1.5-4), Ref. [22], p. 212]

$$ \frac{1}{r} = \frac{1 + e\cos\theta}{p} \triangleq B_1 + B_2\cos\theta \tag{158} $$

where θ is now an angle measured from the periapsis (the point where r is minimum) and referred to as the true anomaly. The constants e and p are known respectively as the eccentricity of the conic section, and the semilatus rectum (or parameter). Differentiation with respect to time brings the constraint equation to the velocity level,

$$ \dot{r} - B_2\sin\theta\, r^2\dot{\theta} = {}^{N}\mathbf{v}^{P_2/P_1} \cdot \hat{\mathbf{a}}_1 - B_2\sin\theta\, r\,{}^{N}\mathbf{v}^{P_2/P_1} \cdot \hat{\mathbf{a}}_2 = 0 \tag{159} $$

or

$$({}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1}) \cdot (\hat{\mathbf{a}}_1 - B_2 \sin\theta\, r\hat{\mathbf{a}}_2) = 0 \tag{160}$$

Inspection of this constraint equation indicates that a constraint force

$$\mathbf{C}_1 = \lambda_1(\hat{\mathbf{a}}_1 - B_2 \sin\theta\, r\hat{\mathbf{a}}_2) \tag{161}$$

must be applied to P_2, and $-\mathbf{C}_1$ must be applied to P_1 if $r\hat{\mathbf{a}}_1$ is to follow a conic section.

The constraint forces needed to impose Kepler's first two laws are considered to be the only forces acting on P_1 and P_2. Hence, Newton's second law dictates that

$$m_2\, {}^N\mathbf{a}^{P_2} = \lambda_1(\hat{\mathbf{a}}_1 - B_2 \sin\theta\, r\hat{\mathbf{a}}_2) + \lambda_2\hat{\mathbf{a}}_2 \tag{162}$$

$$m_1\, {}^N\mathbf{a}^{P_1} = -\lambda_1(\hat{\mathbf{a}}_1 - B_2 \sin\theta\, r\hat{\mathbf{a}}_2) - \lambda_2\hat{\mathbf{a}}_2 \tag{163}$$

where m_1 and m_2 are the respective masses of P_1 and P_2.

At the acceleration level, the constraint dictated by Kepler's second law is obtained by differentiation of Eq. (155) with respect to time,

$$\ddot{A} = \frac{1}{2}(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = 0 \tag{164}$$

Substitution from this result into Eq. (154) leads to

$$({}^N\mathbf{a}^{P_2} - {}^N\mathbf{a}^{P_1}) \cdot \hat{\mathbf{a}}_2 = 0 \tag{165}$$

and, by way of Eqs. (162) and (163), to

$$\frac{1}{m_2}(\lambda_2 - \lambda_1 B_2 \sin\theta\, r) = \frac{1}{m_1}(-\lambda_2 + \lambda_1 B_2 \sin\theta\, r) \tag{166}$$

Consequently,

$$\lambda_2 = \lambda_1 B_2 \sin\theta\, r \tag{167}$$

The resultant constraint force acting on each particle is now seen to have no component in the direction of $\hat{\mathbf{a}}_2$; in other words it is completely in the radial direction, $\hat{\mathbf{a}}_1$. Equations (162) and (163) reduce to

$$m_2\, {}^N\mathbf{a}^{P_2} = \lambda_1\hat{\mathbf{a}}_1, \qquad m_1\, {}^N\mathbf{a}^{P_1} = -\lambda_1\hat{\mathbf{a}}_1 \tag{168}$$

Newton demonstrated the converse of this result. In Proposition I of the *Principia* it is shown that Kepler's second law holds for all motion under the sole influence of any centripetal force whatsoever (Ref. [22], p. 211, and Ref. [81], pp. 418, 425).

Thus far the orbit has not been restricted to be a conic section; imposition of this constraint now leads to an expression for λ_1. After making use of Eq. (155), Kepler's first law at the velocity level expressed in Eq. (159) can be restated as $\dot{r} - 2B_2\dot{A}\sin\theta = 0$. Differentiation with respect to time brings the constraint to the acceleration level.

$$\ddot{r} - 2B_2\dot{A}\cos\theta\,\dot{\theta} = \ddot{r} - 4B_2\dot{A}^2\frac{\cos\theta}{r^2} = 0 \tag{169}$$

where Eq. (155) has been used again in the second step. Substitution from this result into Eq. (154) yields

$$({}^N\mathbf{a}^{P_2} - {}^N\mathbf{a}^{P_1}) \cdot \hat{\mathbf{a}}_1 = \ddot{r} - r\dot{\theta}^2 = 4B_2\dot{A}^2\frac{\cos\theta}{r^2} - r\left(\frac{2\dot{A}}{r^2}\right)^2 = -\frac{4B_1\dot{A}^2}{r^2} = \frac{\lambda_1}{m_2} + \frac{\lambda_1}{m_1} \tag{170}$$

where the penultimate and final steps are made with the help of Eqs. (158) and (168). A solution for λ_1 is now at hand

$$\lambda_1 = -\frac{4B_1\dot{A}^2 m_1 m_2}{(m_1+m_2)r^2} = -\frac{4\dot{A}^2 m_1 m_2}{p(m_1+m_2)r^2} \tag{171}$$

and the constraint force applied to P_2 is given by

$$\mathbf{C} = -\frac{4\dot{A}^2}{p(m_1+m_2)}\frac{m_1 m_2 \hat{\mathbf{a}}_1}{r^2} \tag{172}$$

whereas the constraint force applied to P_1 is simply $-\mathbf{C}$. This result is in conformity with the demonstration in Proposition XI of the *Principia* that elliptical orbits are a consequence of the universal law of gravitation (Ref. [81], p. 429, Ref. [22], p. 212). The necessary gravitational attraction is shown to be proportional to the product of the masses of P_1 and P_2, and inversely proportional to the square of the distance between them.

The fraction $4\dot{A}^2/[p(m_1+m_2)]$ is in fact the universal gravitational constant G. For an elliptical two-body orbit, the identity is demonstrated with the following considerations. First, $p = b^2/a$, where a and b are the semimajor and semiminor axes of the ellipse, respectively. Second, the area of the ellipse, πab, is swept out during one orbital period, T, therefore $\dot{A} = \pi ab/T$. Finally, Kepler's third law states that the square of the period of an elliptical orbit is proportional to the cube of its mean distance from the Sun; that is, $T^2 = 4\pi^2 a^3/[G(m_1+m_2)]$.

CHAPTER 4

A KINEMATICALLY REDUNDANT MANIPULATOR

Motion of a complex mechanical system can be restricted by more than one of the constraints that are treated in isolation in the examples provided in Chapter 3. A multibody manipulator moving a payload onboard an orbiting spacecraft furnishes a case in point. Connections between the bodies can be modeled as revolute joints such as the one discussed in Sec. 3.1.1, and several additional constraints must be considered if one wishes to make the payload follow a compulsory trajectory. A constrained system such as the one just described constitutes an example of substantial intricacy that is used to illustrate application of the material in Chapters 2 and 3. Analysis of the system proceeds to a numerical simulation of motion that entails the calculation of numerous measure numbers of constraint forces and torques needed to bring about prescribed payload motion, and impose configuration constraints associated with selected revolute joints of interest.

A manipulator with seven joints is currently used in construction of the International Space Station in low Earth orbit. As explained in Sec. 4.1, a manipulator is kinematically redundant when seven or more joints are involved, and an algorithm known as a resolved rate law is worked out for determining seven or more joint speeds from six parameters needed to prescribe the motion of a manipulator payload relative to the manipulator's base. The resolved rate law is regarded as a set of equations describing the constraints of prescribed motion at the velocity level, and the equations are inspected to identify motor torques needed to enforce the constraints.

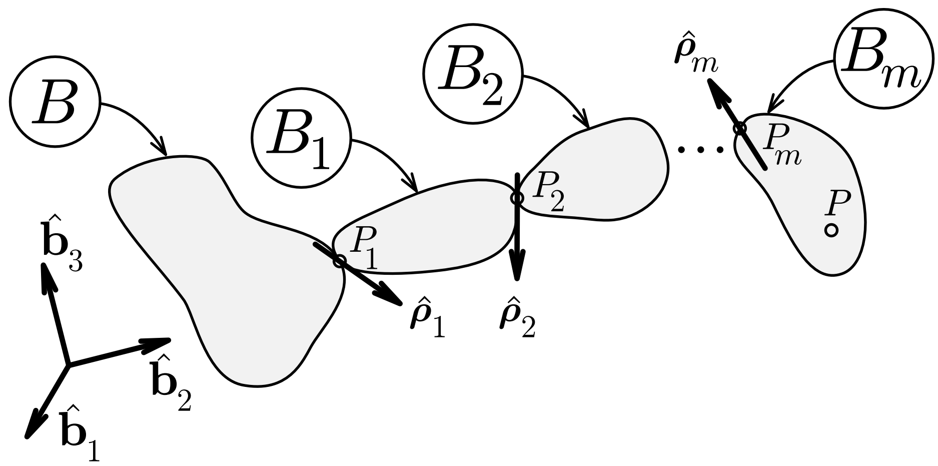


Figure 10: A Serial Remote Manipulator

Section 4.2 contains a detailed description of a model of a system consisting of a base body, a manipulator with seven joints, and a payload. The description includes information needed to create a simulation, namely the way in which the bodies are connected to one another, the way that mass is distributed in each body, and important physical dimensions of the manipulator. Expressions used in prescribing the motion of the payload relative to the base are provided. Interest in the constraints dictated by revolute joints is confined to two of the seven fasteners. Equations describing the restrictions on relative translation at one joint, and the relative change in orientation that is prevented by another joint, are written at the velocity level and inspected by following the example in Sec. 3.1.1, thereby enabling selective identification of constraint force and torque measure numbers and reinforcing the discussion in Sec. 2.5.3. The section concludes with an overview of the dynamical and kinematical differential equations governing the motion of the system, and an introduction of the generalized coordinates, motion variables, and multipliers involved in those equations.

The chapter concludes in Sec. 4.3 with a presentation and discussion of the results from a numerical simulation of a typical manipulator maneuver.

4.1 Resolved Rate Law

A remote manipulator consists of rigid bodies B, B_1, B_2, ..., B_m connected together in a kinematic chain as depicted in Figure 10. Body B is considered as a base to which the manipulator is attached, and B_m is regarded as the payload of the manipulator. Each pair of adjacent bodies is connected by a revolute joint that permits a simple rotation; a unit vector $\hat{\boldsymbol{\rho}}_1$ is fixed in both B and B_1, $\hat{\boldsymbol{\rho}}_2$ is fixed in both B_1 and B_2, and so on. In what follows reference is made to point P_1 fixed in B and in B_1, point P_2 fixed in B_1 and in B_2, etc. Body B_1 has a simple angular velocity in B, ${}^B\boldsymbol{\omega}^{B_1} = u_1\hat{\boldsymbol{\rho}}_1$, body B_2 has a simple angular velocity in B_1, ${}^{B_1}\boldsymbol{\omega}^{B_2} = u_2\hat{\boldsymbol{\rho}}_2$, and so forth.

The motion of the payload, B_m, with respect to the base, B, is most naturally described with the six parameters

$$v_i \triangleq {}^B\mathbf{v}^P \cdot \hat{\mathbf{b}}_i, \qquad \omega_i \triangleq {}^B\boldsymbol{\omega}^{B_m} \cdot \hat{\mathbf{b}}_i \quad (i = 1, 2, 3) \tag{173}$$

where ${}^B\mathbf{v}^P$ is the velocity in B of a point P fixed in B_m, ${}^B\boldsymbol{\omega}^{B_m}$ is the angular velocity of B_m in B, and mutually perpendicular, dextral unit vectors $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, and $\hat{\mathbf{b}}_3$ are fixed in B. In order to direct the motion of each manipulator joint so as to achieve the desired payload motion, one must be able to determine values of joint speeds $u_1, \ldots, u_m$ from specified values of v_i and ω_i ($i = 1, 2, 3$). A solution for the joint speeds is often referred to as a resolved rate law; it entails the solution of a system of six linear equations that is either overdetermined ($m < 6$), square ($m = 6$), or underdetermined ($m > 6$). We concern ourselves here with the last of these situations, in which case the manipulator is called kinematically redundant.

Three of the equations to be solved are obtained right away by invoking the addition theorem for angular velocities.

$$\begin{aligned}\omega_i = {}^B\boldsymbol{\omega}^{B_m} \cdot \hat{\mathbf{b}}_i &= ({}^B\boldsymbol{\omega}^{B_1} + {}^{B_1}\boldsymbol{\omega}^{B_2} + \cdots + {}^{B_{m-1}}\boldsymbol{\omega}^{B_m}) \cdot \hat{\mathbf{b}}_i \\ &= (u_1\hat{\boldsymbol{\rho}}_1 + u_2\hat{\boldsymbol{\rho}}_2 + \cdots + u_m\hat{\boldsymbol{\rho}}_m) \cdot \hat{\mathbf{b}}_i \quad (i = 1, 2, 3)\end{aligned} \tag{174}$$

The remaining three equations are produced by repeated application of the relationship for the velocities of two points fixed on a rigid body.

$$\begin{aligned} v_i &= {}^B\mathbf{v}^P \cdot \hat{\mathbf{b}}_i = ({}^B\boldsymbol{\omega}^{B_1} \times \mathbf{r}^{P_1P_2} + {}^B\boldsymbol{\omega}^{B_2} \times \mathbf{r}^{P_2P_3} + \cdots + {}^B\boldsymbol{\omega}^{B_m} \times \mathbf{r}^{P_mP}) \cdot \hat{\mathbf{b}}_i \\ &= (u_1\hat{\boldsymbol{\rho}}_1 \times \mathbf{r}^{P_1P} + u_2\hat{\boldsymbol{\rho}}_2 \times \mathbf{r}^{P_2P} + \cdots + u_m\hat{\boldsymbol{\rho}}_m \times \mathbf{r}^{P_mP}) \cdot \hat{\mathbf{b}}_i \quad (i = 1, 2, 3) \end{aligned} \tag{175}$$

where $\mathbf{r}^{P_1P_2}$ is the position vector from P_1 to P_2, $\mathbf{r}^{P_2P_3}$ is the position vector from P_2 to P_3, and so forth. The six equations can be expressed in matrix form as

$$v = Ju \tag{176}$$

where v is a 6×1 matrix whose elements are v_1, v_2, v_3, ω_1, ω_2, and ω_3, where u is an $m \times 1$ matrix whose elements are $u_1, \ldots, u_m$, and where the manipulator Jacobian J is a $6 \times m$ matrix whose elements are given by

$$\begin{aligned} J_{ir} &= (\hat{\boldsymbol{\rho}}_r \times \mathbf{r}^{P_rP}) \cdot \hat{\mathbf{b}}_i \quad && (i = 1, 2, 3;\ r = 1, \ldots, m) \\ J_{ir} &= \hat{\boldsymbol{\rho}}_r \cdot \hat{\mathbf{b}}_{i-3} && (i = 4, 5, 6;\ r = 1, \ldots, m) \end{aligned} \tag{177}$$

The usual method of solving Eqs. (176) for u when $m > 6$ is to minimize the sum of the squares of the joint speeds subject to six constraint equations represented by $v - Ju = 0$ (see for example Ref. [17]). The minimum norm solution is then

$$u = [J^{\mathrm{T}}(JJ^{\mathrm{T}})^{-1}]v \triangleq J^{\dagger}v \tag{178}$$

where $J^{\dagger}$ is the well-known Moore-Penrose pseudo inverse matrix, in this case dimensioned $m \times 6$. Henceforth the equations embodied by the resolved rate law (178) are regarded as m constraint equations at the velocity level, $u - J^{\dagger}v = 0$, having the form of Eqs. (49) and expressing a restriction of prescribed motion. In the first such constraint equation, u_1 can be expressed as

$$u_1 = {}^B\boldsymbol{\omega}^{B_1} \cdot \hat{\boldsymbol{\rho}}_1 = ({}^N\boldsymbol{\omega}^{B_1} - {}^N\boldsymbol{\omega}^{B}) \cdot \hat{\boldsymbol{\rho}}_1 \tag{179}$$

where N is a Newtonian reference frame. Inspection of this portion of the constraint equation according to the instructions in Sec. 2.4 reveals that a constraint torque

$\mathbf{T}_1 = \lambda_1 \hat{\boldsymbol{\rho}}_1$ must be applied to B_1, and likewise a constraint torque of $-\mathbf{T}_1$ must be applied to B in order to satisfy the constraint. Similarly, the second constraint equation can be satisfied through the application of $\mathbf{T}_2 = \lambda_2 \hat{\boldsymbol{\rho}}_2$ to B_2, and $-\mathbf{T}_2$ to B_1, and so on. These constraint torques are to be applied by motors at each manipulator joint. The constraint equations can be brought to the acceleration level by differentiation of Eqs. (176) with respect to time, yielding $\dot{v} = \dot{J}u + J\dot{u}$, or

$$\dot{u} + J^{\dagger}(\dot{J}u - \dot{v}) = 0 \tag{180}$$

4.2 Space Station Remote Manipulator System

A six-jointed robotic manipulator used on the Space Shuttle for a number of years served as an inspiration for the design and construction of a seven-jointed robot for the International Space Station (Ref. [70]); both manipulators are employed together in assembly of the Station today. The Space Station Remote Manipulator System (SSRMS) shown in Figure 11 has a high degree of symmetry and either end can be attached to a payload (Ref. [48]). Three joints in close proximity to one another make up the shoulder of the manipulator in the area where it is attached to the Station, and three similar joints make up the wrist near the point of attachment to a payload. Two long booms in the center of the manipulator are joined to form the elbow. In traversing the manipulator from the base outboard to the payload, one encounters the joints in this order: shoulder roll, shoulder yaw, shoulder pitch, elbow pitch, wrist pitch, wrist yaw, and wrist roll. When all joint angles have a value of 0 the configuration of the SSRMS can be described with Table 1 in terms of points and unit vectors depicted in Figure 10. The position of the mass center $B_r^\star$ $(r = 1, \ldots, 6)$ of each body in the manipulator is estimated by assuming that it lies midway between the joints. Points P_7, $B_7^\star$, and P fixed in the payload B_7 are assumed to be coincident for convenience in verification of the manipulator geometry. For simplicity, $B^\star$ and P_1 are also taken to be coincident. The total length of the outstretched manipulator

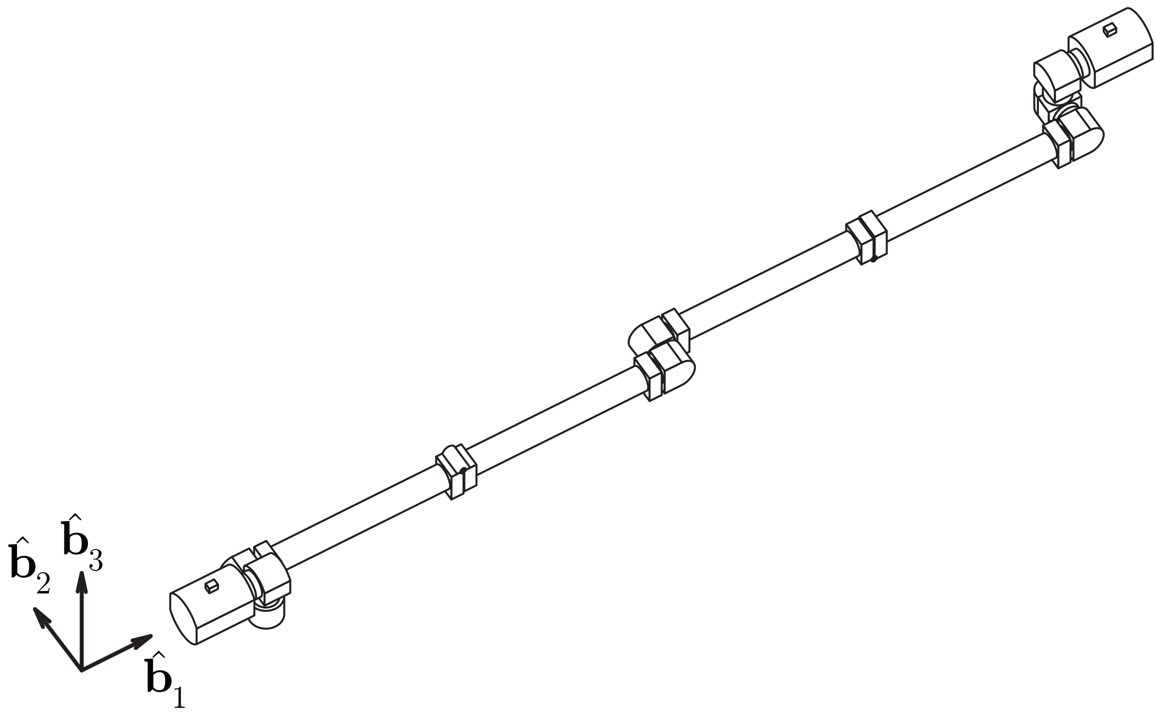


Figure 11: Space Station Remote Manipulator System

is 16.938 m.

A representative mass for the Station, B, the bodies that make up the SSRMS, $B_1, \ldots, B_6$, and the payload, B_7, are recorded in Table 2. The mass of the SSRMS is three orders of magnitude less than that of the Station. The payload's mass is two orders of magnitude less than the Station's mass. The Station's mass distribution can be quantified by the representative central moments and products of inertia listed in the first row of Table 3; when all joint angles have a value of 0, corresponding quantities for the SSRMS and payload are given in the subsequent rows of the table. If $\underline{\mathbf{I}}$ denotes the inertia dyadic of the body of interest with respect to its mass center, then the inertia scalars are defined as $I_{rs} \triangleq \hat{\mathbf{b}}_r \cdot \underline{\mathbf{I}} \cdot \hat{\mathbf{b}}_s$ $(r, s = 1, 2, 3)$, where unit vectors $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, and $\hat{\mathbf{b}}_3$ are fixed in B.

Each revolute joint in the manipulator dictates a configuration constraint as discussed in Sec. 3.1.1. Additional configuration constraints are imposed if the joints are to undergo prescribed motion according to the resolved rate law (178). A simulation

Table 1: SSRMS Configuration With All Joint Angles Equal to Zero

r	$\hat{\boldsymbol{\rho}}_r$	$\mathbf{r}^{P_r B_r^\star}$ (m)			$\mathbf{r}^{B_r^\star P_{r+1}}$ (m)		
		$\hat{\mathbf{b}}_1$	$\hat{\mathbf{b}}_2$	$\hat{\mathbf{b}}_3$	$\hat{\mathbf{b}}_1$	$\hat{\mathbf{b}}_2$	$\hat{\mathbf{b}}_3$
1	$\hat{\mathbf{b}}_1$	0.6795	0	0	0.6795	0	0
2	$\hat{\mathbf{b}}_3$	0	0	-0.3515	0	0	-0.3515
3	$-\hat{\mathbf{b}}_2$	3.5550	0.5690	0	3.5550	0.4750	0
4	$-\hat{\mathbf{b}}_2$	3.5550	0	0	3.5550	0.5690	0
5	$-\hat{\mathbf{b}}_2$	0	0	0.3515	0	0	0.3515
6	$\hat{\mathbf{b}}_3$	0.6795	0	0	0.6795	0	0
7	$\hat{\mathbf{b}}_1$	0	0	0	-	-	-

Table 2: Mass of Base, SSRMS, and Payload

Body	mass (kg)
B	1.92×10^5
B_1	49.62
B_2	6.33
B_3	107.27
B_4	113.25
B_5	6.33
B_6	49.62
B_7	2265.40

Table 3: Mass Distribution of Base, SSRMS, and Payload

Body	Inertia Scalar (kg-m^2)					
	I_{11}	I_{22}	I_{33}	I_{12}	I_{23}	I_{31}
B	1.14×10^7	1.99×10^7	2.52×10^7	-0.68×10^6	-0.89×10^5	0.82×10^6
B_1	3.54	3.54	1.45	0	0	0
B_2	0.18	0.15	0.15	0	0	0
B_3	5.65	569.90	572.81	13.60	0	0
B_4	9.31	638.57	644.01	28.58	0	0
B_5	0.18	0.15	0.15	0	0	0
B_6	3.54	3.54	1.45	0	0	0
B_7	0.58×10^4	4.68×10^4	4.68×10^4	0	0	0

of manipulator motion has been constructed such that a compulsory SSRMS maneuver is carried out through the action of motors that apply at each joint a constraint couple whose torque is parallel to $\hat{\boldsymbol{\rho}}_r$ $(r = 1, \ldots, 7)$; furthermore, the constraint force imposed by the revolute shoulder yaw joint is determined, together with the torque of the constraint couple dictated by the revolute wrist yaw joint. A computer program for carrying out the simulation has been prepared with the aid of AUTOLEV software described in Ref. [45]. The Station B is permitted six degrees of freedom in a Newtonian reference frame N. The weightlessness of orbital motion is represented in the simulation by absence of gravitational or other external forces. The action of devices that control the orientation of B in N, such as control moment gyroscopes, flywheels, or reaction control thrusters, is not modeled in the simulation.

A prescription for the SSRMS maneuver is written as follows. First, the path traveled by point P in B during an interval $\tau_0 \leq t \leq \tau_1$ is taken to be a straight line. It is convenient to define the total distance Q traveled during this interval, together with a unit vector $\hat{\boldsymbol{\mu}}$ fixed in B and parallel to this line,

$$Q \triangleq |\mathbf{r}^{P_1 P}(\tau_1) - \mathbf{r}^{P_1 P}(\tau_0)|, \qquad \hat{\boldsymbol{\mu}} \triangleq \frac{\mathbf{r}^{P_1 P}(\tau_1) - \mathbf{r}^{P_1 P}(\tau_0)}{Q} \tag{181}$$

The magnitude q of the displacement of P in B is taken to be the following function of t,

$$q(t) = \frac{Q}{\tau_1 - \tau_0}(t - \tau_0) - \frac{Q}{2\pi} \sin \frac{2\pi(t - \tau_0)}{\tau_1 - \tau_0} \tag{182}$$

so that its derivative with respect to t is given by

$$\dot{q}(t) = \frac{Q}{\tau_1 - \tau_0} \left[1 - \cos \frac{2\pi(t - \tau_0)}{\tau_1 - \tau_0} \right] \tag{183}$$

The velocity of P in B is then written as ${}^B\mathbf{v}^P = \dot{q}(t)\hat{\boldsymbol{\mu}}$ and, for simplicity, it is assumed that during the maneuver the angular velocity of the payload in B is zero, ${}^B\boldsymbol{\omega}^{B_7} \equiv \mathbf{0}$. Hence, the elements of the array v in Eq. (176) are given by

$$v_i = \dot{q}(t)\hat{\boldsymbol{\mu}} \cdot \hat{\mathbf{b}}_i, \qquad \omega_i \triangleq {}^B\boldsymbol{\omega}^{B_7} \cdot \hat{\mathbf{b}}_i = 0 \quad (i = 1, 2, 3) \tag{184}$$

The simulation is performed with $\tau_0 = 0$, at which time the seven joint angles have the values 99.0806, −19.6069, −36.2539, 95.5385, 57.0514, 4.93519, and 58.7751 degrees corresponding to $\mathbf{r}^{P_1P} = 9.60228\hat{\mathbf{b}}_1 - 1.906517\hat{\mathbf{b}}_2 - 1.4187\hat{\mathbf{b}}_3$ m. The simulation ends at $\tau_1 = 30$ sec, at which time the displacement of P is required to be $\mathbf{r}^{P_1P}(\tau_1) - \mathbf{r}^{P_1P}(\tau_0) = \sqrt{5}\hat{\mathbf{b}}_1 - 2\hat{\mathbf{b}}_2 + 1\hat{\mathbf{b}}_3$ m.

As discussed in Sec. 4.1, seven constraint equations at the velocity level are expressed with the resolved rate law (178), and inspection of those equations reveals that body B_s is therefore subject to a constraint couple (exerted by a motor) whose torque is given by $\mathbf{T}_s = \lambda_s \hat{\boldsymbol{\rho}}_s$; a constraint couple of torque $-\mathbf{T}_s$ is applied to the body that is inboard of B_s $(s = 1, \ldots, 7)$.

If point P_2 is regarded as fixed in B_2 and coincident with a point Q_2 fixed in B_1, the revolute shoulder yaw joint prevents the two points from separating. This configuration constraint can be expressed at the velocity level by the three equations

$$({}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{Q_2}) \cdot \hat{\mathbf{d}}_{s-7} = u_{s+6} = 0 \qquad (s = 8, 9, 10) \tag{185}$$

where ${}^N\mathbf{v}^{P_2}$ and ${}^N\mathbf{v}^{Q_2}$ are the velocities in N of P_2 and Q_2 respectively. Unit vectors $\hat{\mathbf{d}}_1$, $\hat{\mathbf{d}}_2$, and $\hat{\mathbf{d}}_3$ are fixed in B_2 and aligned with $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, and $\hat{\mathbf{b}}_3$ when the values of the shoulder roll and shoulder yaw joint angles vanish. Inspection of these constraint equations indicates that a constraint force

$$\mathbf{C} = \lambda_8 \hat{\mathbf{d}}_1 + \lambda_9 \hat{\mathbf{d}}_2 + \lambda_{10} \hat{\mathbf{d}}_3 \tag{186}$$

is applied by B_1 to B_2 at P_2 and, by the same token, a constraint force $-\mathbf{C}$ is applied by B_2 to B_1 at Q_2.

Let $\hat{\mathbf{h}}_1$, $\hat{\mathbf{h}}_2$, and $\hat{\mathbf{h}}_3$ be a set of dextral, orthogonal unit vectors fixed in B_6 such that they have the same directions as $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, and $\hat{\mathbf{b}}_3$ respectively when all SSRMS joint angles with the possible exception of the wrist roll joint have a value of 0. Furthermore, $\hat{\mathbf{h}}_3$ is also fixed in B_5 and is identical to unit vector $\hat{\boldsymbol{\rho}}_6$. The revolute wrist yaw joint prevents a change in the relative orientation of B_6 and B_5 in any direction that lies in the plane formed by $\hat{\mathbf{h}}_1$ and $\hat{\mathbf{h}}_2$; at the velocity level, this configuration constraint can be expressed with the relationships

$$({}^N\boldsymbol{\omega}^{B_6} - {}^N\boldsymbol{\omega}^{B_5}) \cdot \hat{\mathbf{h}}_{s-10} = u_{s+6} = 0 \qquad (s = 11, 12) \tag{187}$$

Inspection of the two constraint equations reveals that a constraint torque

$$\mathbf{T} = \lambda_{11}\hat{\mathbf{h}}_1 + \lambda_{12}\hat{\mathbf{h}}_2 \tag{188}$$

is exerted by B_5 on B_6, and a constraint torque $-\mathbf{T}$ is applied by B_6 onto B_5.

The software AUTOLEV is used to form 18 dynamical differential equations of motion

$$\mathcal{M}\dot{u} = f + \alpha^{\mathrm{T}}\lambda \tag{189}$$

as well as requisite kinematical differential equations, and write a computer program to effect a numerical solution of these equations. The elements of the column array $\dot{u}$ are the time derivatives of $u_1, \ldots, u_{18}$. The first three motion variables, u_1, u_2, and u_3, are used to characterize the velocity in N of the mass center $B^\star$ of B, ${}^N\mathbf{v}^{B^\star} = u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2 + u_3\hat{\mathbf{n}}_3$, where unit vectors $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$, and $\hat{\mathbf{n}}_3$ are fixed in N and have the same directions as $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, and $\hat{\mathbf{b}}_3$ when $t = 0$. The motion variables u_4, u_5, and u_6 describe the angular velocity of B in N, ${}^N\boldsymbol{\omega}^B = u_4\hat{\mathbf{b}}_1 + u_5\hat{\mathbf{b}}_2 + u_6\hat{\mathbf{b}}_3$. The SSRMS joint speeds $u_7, \ldots, u_{13}$ are prescribed via the resolved rate law and are associated, respectively, with the shoulder roll, shoulder yaw, shoulder pitch, elbow pitch, wrist pitch, wrist yaw, and wrist roll joints. The quantities $u_{14}, \ldots, u_{18}$ characterize motion that the system cannot in fact possess because of the presence of the revolute shoulder and wrist yaw joints; their actual values and the values of their time derivatives are 0 according to Eqs. (185) and (187). The elements of the column array λ are $\lambda_1, \ldots, \lambda_{12}$, where $\lambda_1, \ldots, \lambda_7$ are measure numbers of the motor torques needed to move the SSRMS payload in the prescribed manner; λ_8, λ_9, and λ_{10}, as well as λ_{11} and λ_{12}, have the same meanings as in Eqs. (186) and (188). The matrix α is dimensioned 12×18; a 12×6 matrix of zeros is contained within the first 6 columns of α whereas the 12×12 identity matrix lies in the final 12 columns of α. The quantities u_{s+6} and $\dot{u}_{s+6}$ $(s=1,\ldots,12)$ are specified by constraint equations; hence, Eqs. (189) are solved for the 18 unknowns $\dot{u}_r$ $(r=1,\ldots,6)$ and λ_s $(s=1,\ldots,12)$.

The configuration in N of the system composed of the base, SSRMS, and payload is specified with generalized coordinates $q_1,\ldots,q_{13}$, the first three of which describe the position from a point O fixed in N to $B^\star$, $\mathbf{r}^{OB^\star}=q_1\hat{\mathbf{n}}_1+q_2\hat{\mathbf{n}}_2+q_3\hat{\mathbf{n}}_3$. Generalized coordinates q_4, q_5, and q_6 are angles that belong to a body-three, 1-2-3 rotation sequence (Ref. [46], p. 423) and describe the orientation of B in N. The SSRMS joint angles are denoted by $q_7,\ldots,q_{13}$. Kinematical differential equations are given by $\dot{q}_r=u_r$ for $r=1,2,3,7,\ldots,13$; equations that relate $\dot{q}_4$, $\dot{q}_5$, and $\dot{q}_6$ to u_4, u_5, and u_6 can be obtained by referring to p. 427 of Ref. [46]:

$$\dot{q}_4=(u_4\cos q_6-u_5\sin q_6)/\cos q_5 \tag{190}$$

$$\dot{q}_5=u_4\sin q_6+u_5\cos q_6 \tag{191}$$

$$\dot{q}_6=(-u_4\cos q_6+u_5\sin q_6)\sin q_5/\cos q_5+u_6 \tag{192}$$

4.3 Simulation Results

Results of a simulation of an SSRMS maneuver are presented in what follows. Initial values of q_r and u_r $(r=1,\ldots,6)$ are 0; initial values of the joint angles are as stated previously, $q_7=99.0806°$, $q_8=-19.6069°$, $q_9=-36.2539°$, $q_{10}=95.5385°$, $q_{11}=57.0514°$, $q_{12}=4.93519°$, and $q_{13}=58.7751°$. Numerical integration of differential equations is performed with a variable step-size Runge-Kutta algorithm, with tolerances for relative and absolute error set respectively to 1×10^{-7} and 1×10^{-8}.

Time histories of $\mathbf{r}^{OB^\star}$ and the orientation of B in N are displayed in Figure 12. The magnitude of displacement of $B^\star$ in N is approximately a factor of 100 less than the prescribed magnitude of displacement of P in B, $\sqrt{10}$ m; this is reasonable in that the masses of B_7 and B have the same proportion. Likewise, the slight change

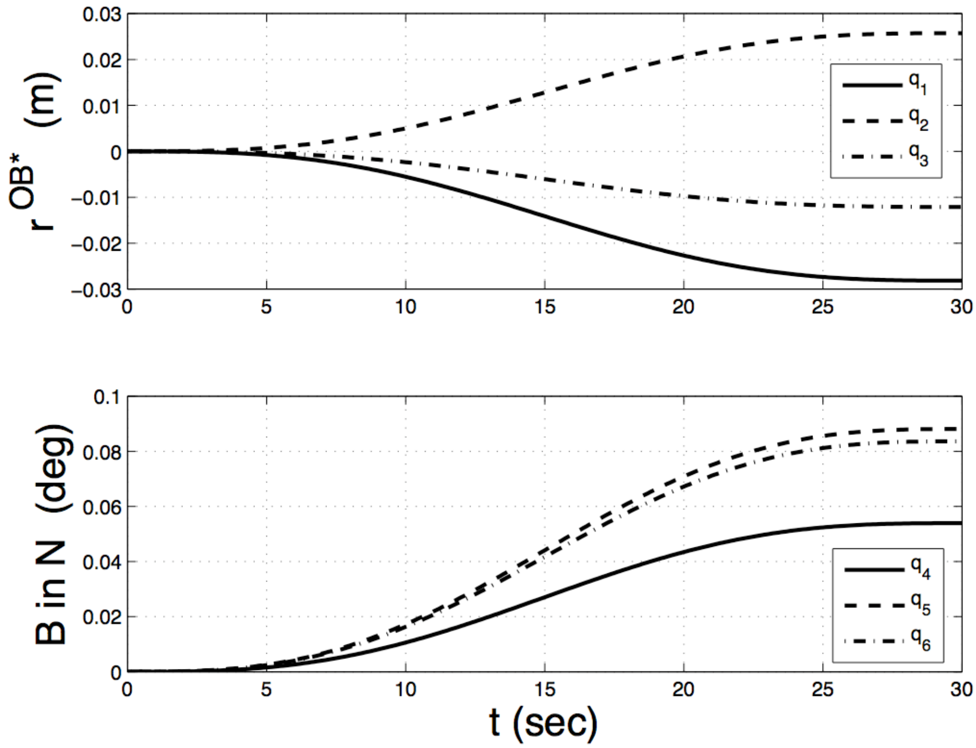


Figure 12: Position and Orientation of B

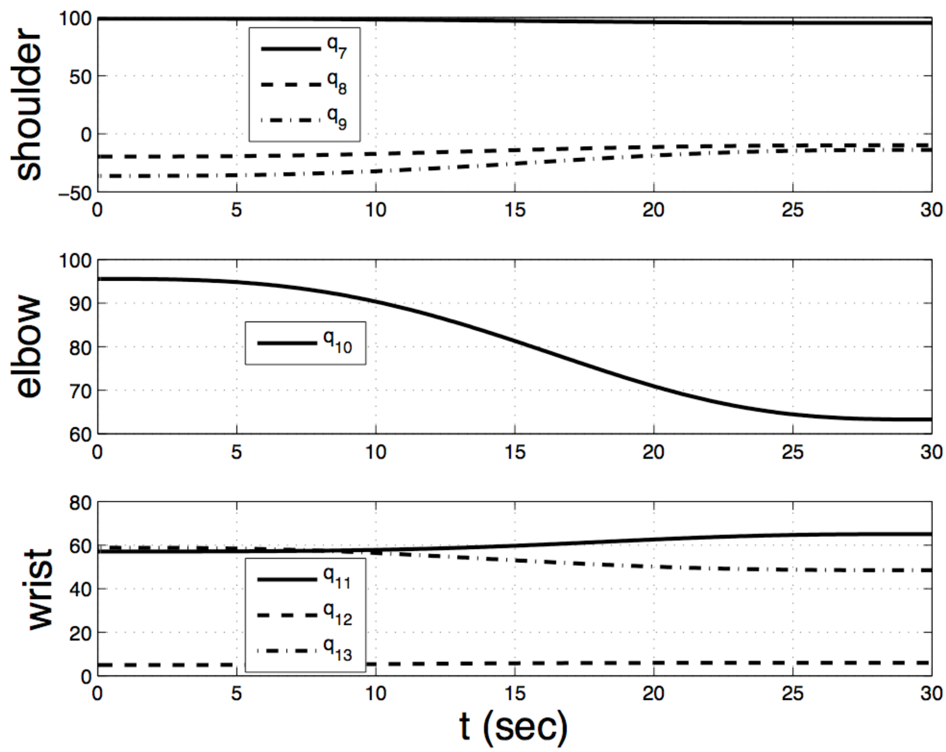


Figure 13: SSRMS Joint Angles (deg)

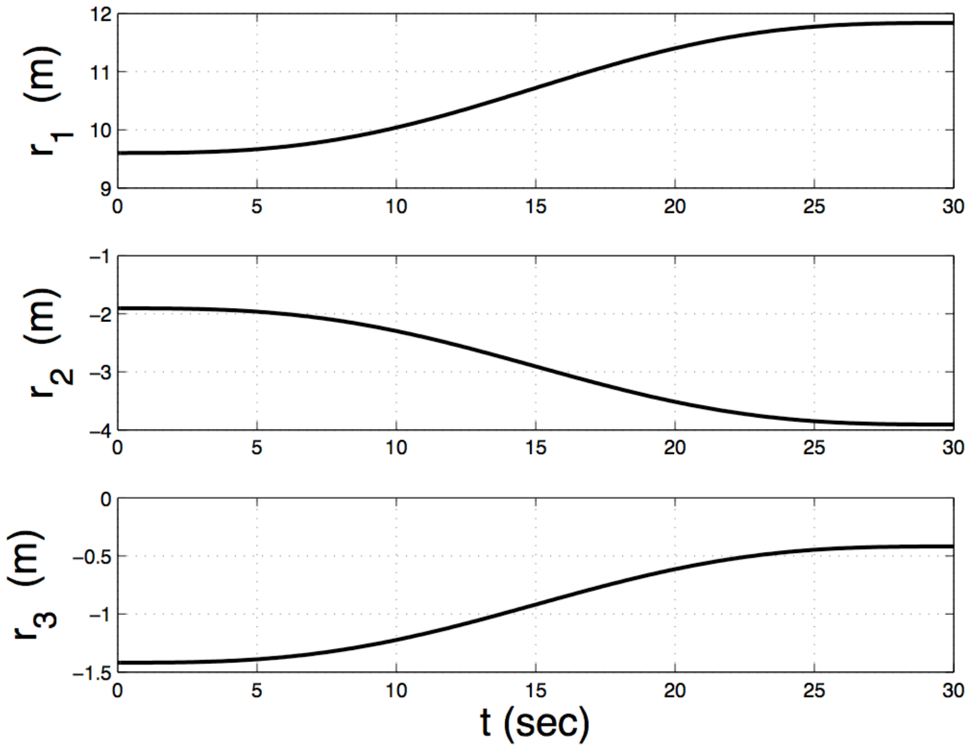


Figure 14: Position of Payload

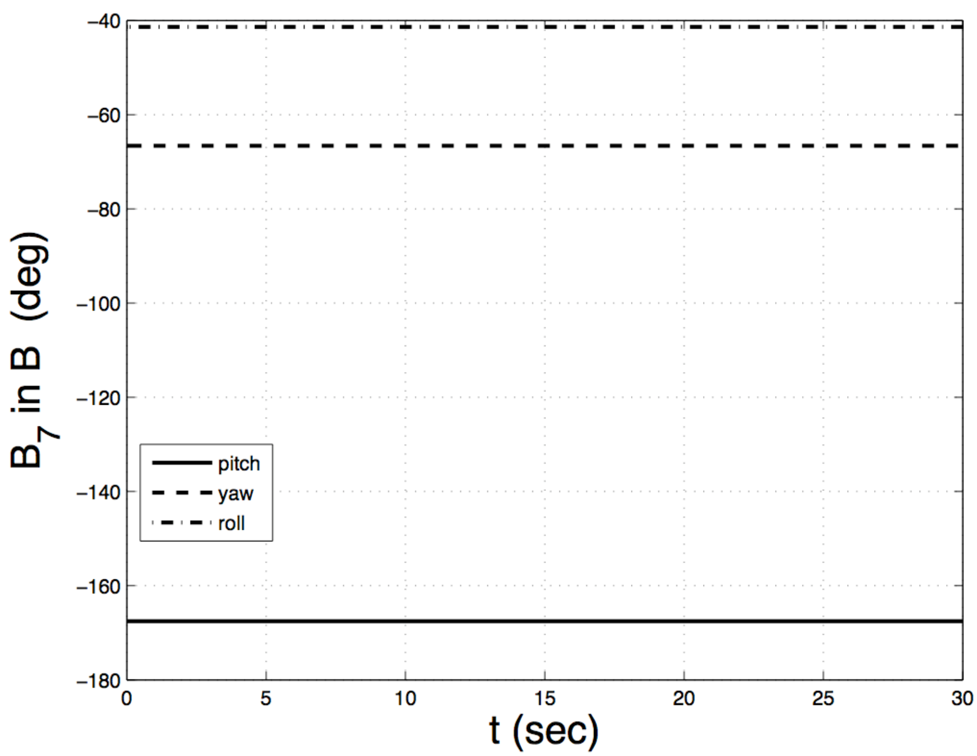


Figure 15: Orientation of Payload

of orientation of B in N is to be expected in view of the inertia scalars of B that are quite large in comparison to the mass distribution of the SSRMS and payload. The prescriptions in Eqs. (182) and (183) include a smooth transition during the interval $\tau_0 \leq t \leq \tau_1$, and absence of a first and second time derivative at the interval's boundaries; these properties are present in the curves of Figure 12 and most of the figures that follow.

Figure 13 shows the change with time of each SSRMS joint angle, all of which remain within the required limits of ± 270 deg. The corresponding change in payload position is plotted in Figure 14, where $r_i \triangleq \mathbf{r}^{P_1 P} \cdot \hat{\mathbf{b}}_i$ $(i = 1, 2, 3)$. It is clear that the point P is displaced in B according to prescription; that is, r_1 increases by $\sqrt{5}$ m, r_2 decreases by 2 m, and r_3 increases by 1 m. The differences between the actual and prescribed values of the three quantities are, respectively, 2×10^{-6}, 0, and 4×10^{-7} m. The prescription in Eqs. (184) calls for ${}^B\boldsymbol{\omega}^{B_7} = \mathbf{0}$; thus, there should be no change in orientation of B_7 relative to B during the simulation. This expectation is confirmed in Figure 15, where pitch, yaw, and roll are angles that belong to a body-three, 2-3-1 rotation sequence and describe the orientation of B_7 in B.

Time histories of the SSRMS joint speeds are illustrated in Figure 16. It can be seen that the manipulator is at rest in B at the beginning and at the end of the simulation. The magnitudes of the shoulder, elbow, and wrist joint speeds remain below their respective operational limits of 2.29, 3.21, and 4.76 deg/sec. The joint speeds and the manipulator Jacobian are used together with Eqs. (176) to obtain the curves of the actual values of v_1, v_2, v_3, ω_1, ω_2, and ω_3 contained in Figure 17. Curves of the prescribed values of v_1, v_2, v_3 obtained from the first three of Eqs. (184) are indistinguishable from those in Figure 17. The magnitude of ${}^B\mathbf{v}^P$ reaches a maximum of 0.21 m/sec at $t = 15$ sec, midway through the maneuver. The final three of Eqs. (184) indicate that $\omega_i(t) = 0$ $(i = 1, 2, 3)$ during the obligatory motion; Figure 17 shows that none of these measure numbers of ${}^B\boldsymbol{\omega}^{B_7}$ grows larger in magnitude than

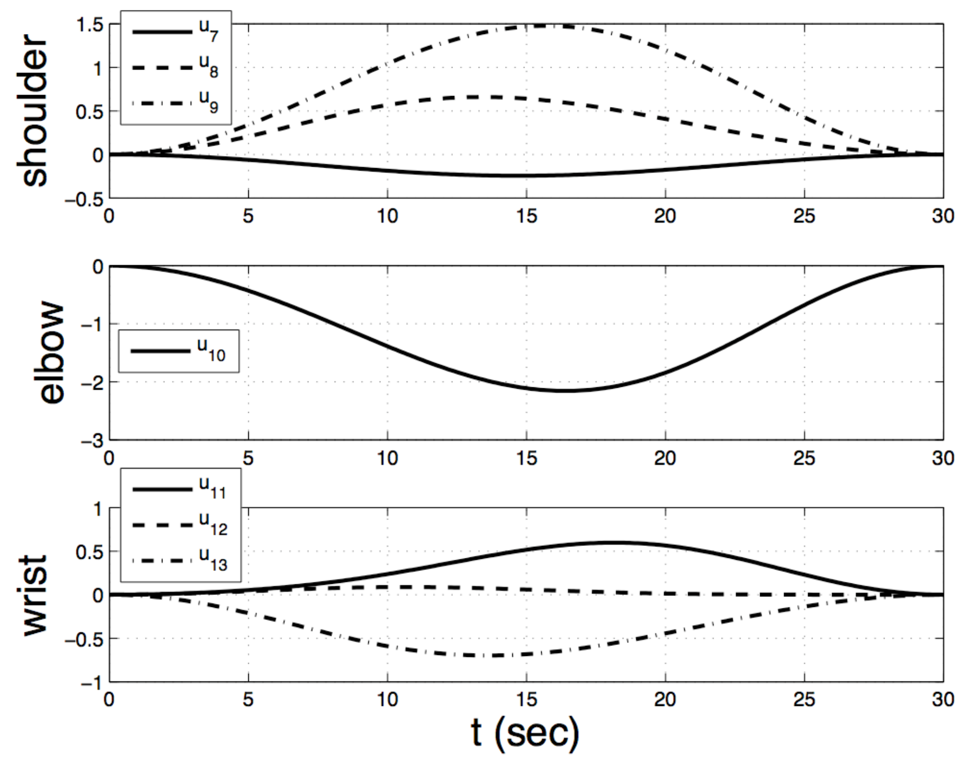


Figure 16: SSRMS Joint Speeds (deg/sec)

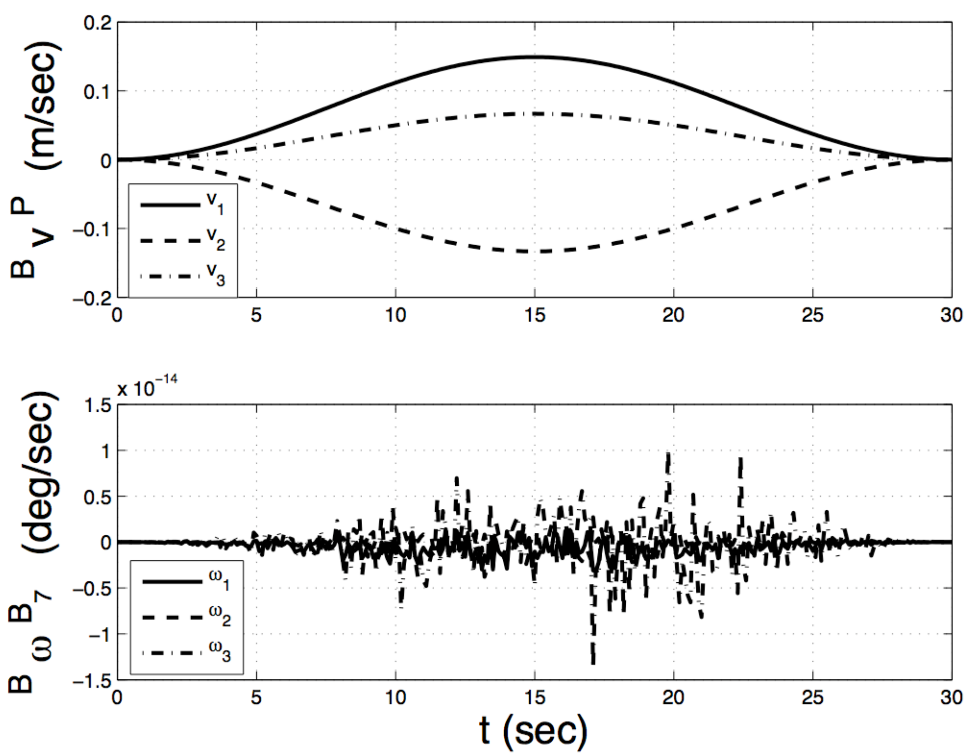


Figure 17: Payload Motion

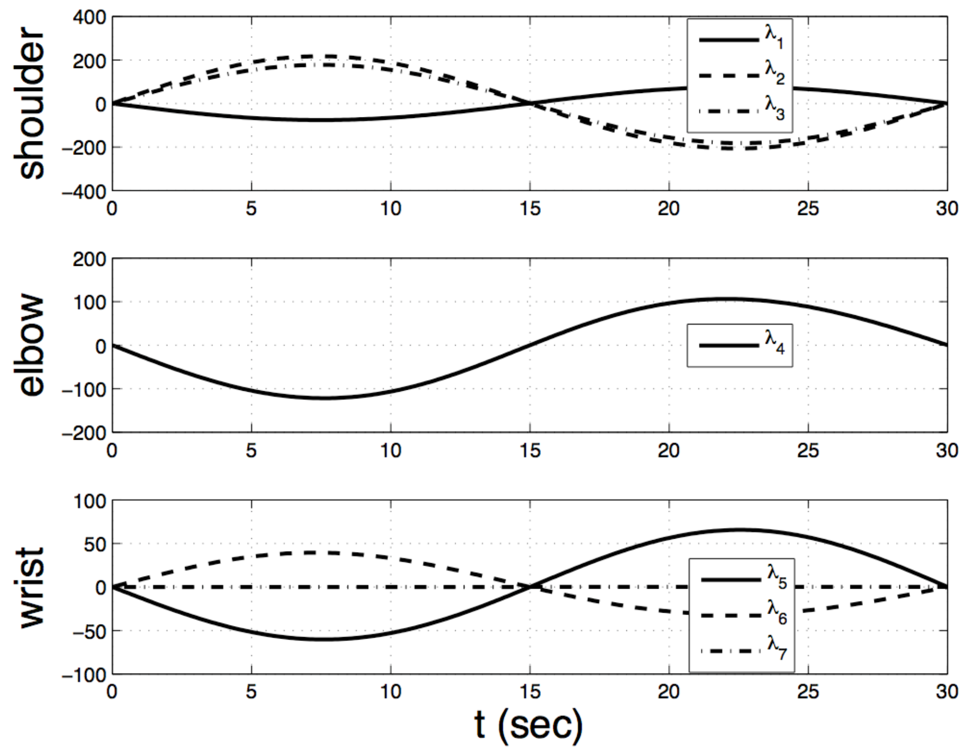


Figure 18: SSRMS Joint Motor Torques (Nm)

1.5×10^{-14} deg/sec during the simulation.

The SSRMS joint motor torques required to compel the manipulator to move the payload according to the prescribed schedule are described in Figure 18. Although it is not evident at this scale, the wrist roll joint motor torque measure number, λ_7, is nearly sinusoidal with an amplitude of 6.7×10^{-2} Nm. All motor torque magnitudes are well within 1044 Nm, a minimum value contained in preliminary design specifications reported in Ref. [48], as well as the value of 2545 Nm reported in Ref. [26].

Figure 19 provides time histories of measure numbers of the constraint force imposed by the revolute shoulder yaw joint. Each measure number is less in absolute value that 40 N during the maneuver; these are low values in contrast to a force of more than 2×10^5 N that would be required to support the payload on the Earth's surface. This result helps to explain why the SSRMS is such a light mechanism in comparison to a terrestrial manipulator designed to move a payload of similar mass.

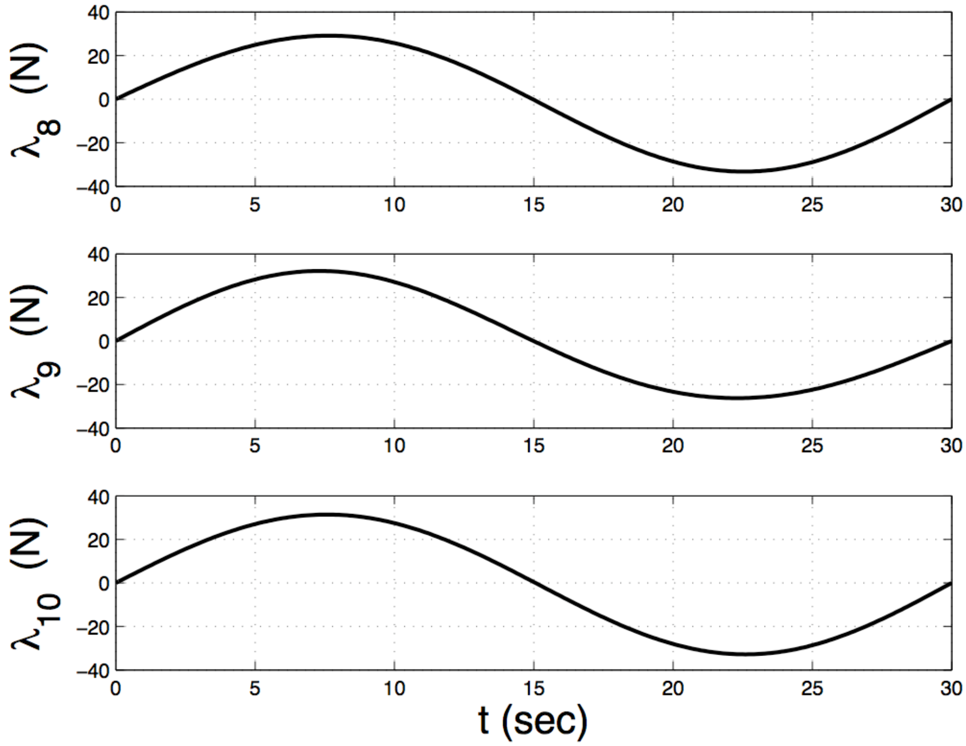


Figure 19: Shoulder Yaw Joint Constraint Force

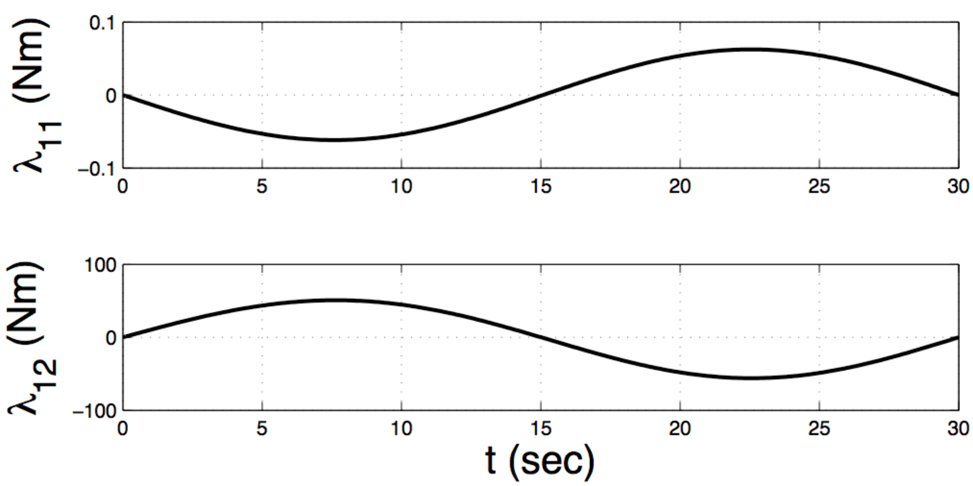


Figure 20: Wrist Yaw Joint Constraint Torque

Measure numbers of the constraint torque dictated by the revolute wrist yaw joint in two directions perpendicular to the axis of the joint are shown in Figure 20. The amplitude of λ_{12} is on the order of the motor torque measure numbers in Figure 18, whereas the amplitude of λ_{11} is quite small. These results are reasonable in view of the vigorous exertion in the wrist pitch joint motor (λ_5) displayed in Figure 18, and the comparatively lethargic activity in the wrist roll joint motor (λ_7).

In conclusion, the process set forth in Sec. 2.4 for inspecting constraint equations at the velocity level permits identification of constraint torques that must be exerted by manipulator motors in order for the payload to move on a prescribed trajectory; application of the calculated torques places the payload within 2×10^{-6} m of the specified final position. The process of inspection is also applied in a selective manner to obtain information about certain constraint forces and torques associated with revolute joints of interest.

CHAPTER 5

NONLINEAR NONHOLONOMIC CONSTRAINT EQUATIONS

The nonholonomic constraint equations dealt with in Chapters 2 and 3 are linear in the motion variables. Although many motion constraints encountered in practice can be described by such equations, one may consider a more general form that is nonlinear in the motion variables, $f(q_1, \ldots, q_n, u_1, \ldots, u_n, t) = 0$. Roberson and Schwertassek (Ref. [60], p. 96) note that all known motion constraints imposed on purely mechanical systems can be expressed with relationships that are linear in velocity variables. In Ref. [8] Bajodah et al. review some of the literature dealing with nonlinear nonholonomic constraint equations and consider it important to study them because they can arise in connection with servo-constraints or program constraints when a control system enters the picture. As explained in Refs. [7] and [19], such constraints are enforced by application of control forces as opposed to the forces present when bodies and particles come into contact with one another, as is the case with classical, passive constraints that have been studied so far in this work.

Methods for dealing with nonlinear nonholonomic constraint equations are frequently illustrated by applying them to the Appell-Hamel mechanism. It is studied and discussed, for example, in Refs. [5], [8], [37], [40], [51], [54], and [83]; however, it is known that the constraints imposed on this mechanical system can be expressed with linear relationships. In Refs. [85] and [86], Zekovich offers several examples of systems in which the constraints can be described with nonlinear nonholonomic constraint equations. Each example involves planar motion of two particles connected by a massless rigid rod or by a massless prismatic joint. Sharp-edged blades are attached in various ways so as to cause the velocities of the particles in an inertial reference frame N to be parallel, equal in magnitude, or perpendicular. In what follows it is shown that the associated constraints can in fact be expressed with linear nonholonomic equations, but that the restrictions on the velocities are inherently nonlinear when the particles are not physically connected and the constraints are dictated by means other than the blades.

The literature contains additional instances of nonlinear nonholonomic constraint equations. Another case of planar motion of two particles with parallel velocities, which serves as an example in Refs. [40], [47], [71], and [84], is brought about with a device proposed by Benenti in Ref. [16]. Benenti's mechanism consists of six rigid rods, one revolute joint, two blades, and at least eight (if not sixteen) prismatic joints. Jankowski provides two examples in Ref. [41] involving a single particle moving in a vertical plane subject to a uniform gravitational field and air resistance; the magnitude of the particle's velocity in N, or the magnitude of the acceleration in N, must match a prescribed time history. References [47], [51], [75], and [76] include an example proposed by Appell in which the velocity ${}^N\mathbf{v}^P$ in N of a particle P must satisfy the relationship ${v_3}^2 = a^2({v_1}^2 + {v_2}^2)$, where a is a constant and v_r are the dot products of ${}^N\mathbf{v}^P$ with a set of right-handed, mutually perpendicular unit vectors $\hat{\mathbf{n}}_r$ fixed in N $(r = 1, 2, 3)$. Control of an inverted pendulum constitutes an example studied in Refs. [47] and [71]. A thin rigid rod moves in a vertical plane in the presence of a uniform gravitational field, with the lower end of the rod always in contact with a horizontal line. The system is referred to as Marle's servomechanism; as proposed in Ref. [51], an actuator controls the horizontal displacement of the rod's lower end according to some control law in order to keep the rod vertical. An earlier paper by Huston and Passerello (Ref. [38]) considers the more general case of balancing a pole not confined to a vertical plane by controlling the position of the pole's lower end that remains in contact with a horizontal plane. A multiple pendulum undergoing planar motion in a uniform gravitational field is considered in Ref. [56]; with the angle between the kth rod and the vertical denoted by θ_k, it is proposed that the motion of each revolute joint obey the constraint equation $\theta_k = C(\dot{\theta}_{k-1})^3$, where C is a given constant.

In what follows, a treatment of nonlinear nonholonomic constraint equations is undertaken in Sec. 5.1 for a generic system of particles; the results are applicable whether or not a subset of particles makes up a rigid body. Two new methods of dealing with nonlinear nonholonomic constraint equations are developed. The first method produces dynamical equations of motion in which evidence of the constraint forces is present, whereas constraint forces are not in evidence in the equations of motion obtained with the second method. Both methods are applied in four examples, the first three of which involve two particles whose velocities must be, respectively, parallel, equal in magnitude, and perpendicular. The final demonstration is performed with Appell's particle. In Sec. 5.2 the two methods are adapted to the special case in which a system of particles contains a rigid body. The chapter ends in Sec. 5.3 with a look at the question of whether or not an energy integral can be used to advantage as a nonlinear nonholonomic constraint equation; it is concluded that an integral of the motion cannot be so employed.

5.1 A System of Particles

It is instructive to recall Eqs. (37) are, in general, nonlinear holonomic constraint equations at the position level that, when expressed at the velocity level, are linear in the velocity vectors as shown in Eqs. (39) or, what is the same, linear in the motion variables as indicated in Eqs. (49). Similarly, nonlinear nonholonomic constraint equations, when expressed at the acceleration level, are linear in the acceleration vectors; that is, they have the form of Eqs. (43) and are thus linear in the time derivatives of motion variables when written in scalar form. Two important conclusions follow from these observations. First, forces needed to satisfy nonlinear nonholonomic constraint equations can be formed with the approach described in Sec. 2.1.4. Second, partial accelerations can be used in place of partial velocities to eliminate the constraint forces from equations of motion in which they appear.

Suppose that a simple nonholonomic system S is made up of particles $P_1, \ldots, P_\nu$. The configuration of S in a Newtonian reference frame N is described by generalized coordinates $q_1, \ldots, q_n$, and the motion of S is characterized by motion variables $u_1, \ldots, u_p$. Suppose further that S is subject to l nonlinear nonholonomic constraint equations

$$h_s({}^N\mathbf{v}^{P_1}, \ldots, {}^N\mathbf{v}^{P_\nu}, t) = 0 \qquad (s = 1, \ldots, l) \tag{193}$$

In this case S is referred to as a *complex nonholonomic system*. Differentiation of these relationships with respect to t in N yields

$$\sum_{i=1}^{\nu} {}^N\mathbf{a}^{P_i} \cdot \mathbf{W}_{is} + Z_s = 0 \qquad (s = 1, \ldots, l) \tag{194}$$

where $\mathbf{W}_{is}$ are vector functions of $q_1, \ldots, q_n, u_1, \ldots, u_p$ and t in N, and Z_s are scalar functions of the same variables. According to the material presented in Sec. 2.1.4, constraint forces obtained by inspecting these relationships are given by

$$\mathbf{C}_{is} = \lambda_s \mathbf{W}_{is} \qquad (i = 1, \ldots, \nu;\ s = 1, \ldots, l) \tag{195}$$

Dynamical equations of motion to which $\mathbf{C}_{is}$ do contribute are given by

$$\begin{aligned}\tilde{F}_r + \tilde{F}_r^\star &= \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \left(\mathbf{R}_i - m_i\, {}^N\mathbf{a}^{P_i}\right) \\ &= \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{v}}_r^{P_i} \cdot \left(\mathbf{f}_i + \sum_{s=1}^{l} \lambda_s \mathbf{W}_{is} - m_i\, {}^N\mathbf{a}^{P_i}\right) = 0 \qquad (r = 1, \ldots, p)\end{aligned} \tag{196}$$

Equations (196) together with Eqs. (194) furnish the number of relationships needed to solve for the unknown quantities $\dot{u}_1, \ldots, \dot{u}_p, \lambda_1, \ldots, \lambda_l$. A reduced or minimal set of dynamical equations to which $\mathbf{C}_{is}$ do not contribute are given by

$$\begin{aligned}\approx{F}_r + \approx{F}_r^\star &= \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \left(\mathbf{f}_i + \sum_{s=1}^{l} \lambda_s \mathbf{W}_{is} - m_i \, {}^N\mathbf{a}^{P_i}\right) \\ &= \sum_{i=1}^{\nu} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \left(\mathbf{f}_i - m_i \, {}^N\mathbf{a}^{P_i}\right) = 0 \qquad (r = 1, \dots, c) \end{aligned} \tag{197}$$

where

$$c \triangleq p - l \tag{198}$$

is the number of degrees of freedom of S in N. When speaking of $\approx{F}_r$ and $\approx{F}_r^\star$ it is convenient to refer to them, respectively, as the rth nonholonomic generalized active force and the rth nonholonomic generalized inertia force, but the double tilde notation should be used to indicate they have been formed with ${}^N\tilde{\mathbf{a}}_r^{P_i}$, the rth *nonholonomic partial acceleration* of P_i in N, rather than ${}^N\tilde{\mathbf{v}}_r^{P_i}$. Instructions for obtaining nonholonomic partial accelerations are now given, and their role in eliminating the multipliers from Eqs. (197) is discussed presently.

The acceleration of P_i in N can be written uniquely as

$${}^N\mathbf{a}^{P_i} = \sum_{r=1}^{p} {}^N\mathbf{a}_r^{P_i} \, \dot{u}_r + {}^N\mathbf{a}_t^{P_i} \qquad (i = 1, \dots, \nu) \tag{199}$$

and also uniquely as

$${}^N\mathbf{a}^{P_i} = \sum_{r=1}^{c} {}^N\tilde{\mathbf{a}}_r^{P_i} \, \dot{u}_r + {}^N\tilde{\mathbf{a}}_t^{P_i} \qquad (i = 1, \dots, \nu) \tag{200}$$

The first of these expressions can be obtained from Eqs. (2.14.4) of Ref. [44] by differentiation with respect to t in N, in which case the partial acceleration ${}^N\mathbf{a}_r^{P_i}$ is seen to be identical to the nonholonomic partial velocity of P_i in N,

$${}^N\mathbf{a}_r^{P_i} \triangleq {}^N\tilde{\mathbf{v}}_r^{P_i} \qquad (i = 1, \dots, \nu; \; r = 1, \dots, p) \tag{201}$$

and the acceleration remainder ${}^N\mathbf{a}_t^{P_i}$ is defined to be

$${}^N\mathbf{a}_t^{P_i} \triangleq \sum_{r=1}^{p} \left(\frac{{}^N d}{dt} \, {}^N\tilde{\mathbf{v}}_r^{P_i}\right) u_r + \frac{{}^N d}{dt} \, {}^N\tilde{\mathbf{v}}_t^{P_i} \qquad (i = 1, \dots, \nu) \tag{202}$$

Substitution from Eqs. (199) into (194) gives

$$\sum_{r=1}^{p}\left(\sum_{i=1}^{\nu} {}^{N}\mathbf{a}_r^{P_i}\cdot\mathbf{W}_{is}\right)\dot{u}_r + \sum_{i=1}^{\nu} {}^{N}\mathbf{a}_t^{P_i}\cdot\mathbf{W}_{is} + Z_s = 0 \qquad (s=1,\ldots,l) \tag{203}$$

The coefficients of $\dot{u}_r$ and the remaining terms can be abbreviated respectively by means of two definitions,

$$\alpha_{sr} \triangleq \sum_{i=1}^{\nu} {}^{N}\mathbf{a}_r^{P_i}\cdot\mathbf{W}_{is} \qquad (s=1,\ldots,l;\ r=1,\ldots,p) \tag{204}$$

and

$$\gamma_s \triangleq Z_s + \sum_{i=1}^{\nu} {}^{N}\mathbf{a}_t^{P_i}\cdot\mathbf{W}_{is} \qquad (s=1,\ldots,l) \tag{205}$$

These definitions allow Eqs. (203) to be rewritten in a form that is linear in the time derivatives of the motion variables

$$\sum_{r=1}^{p}\alpha_{sr}\dot{u}_r + \gamma_s = 0 \qquad (s=1,\ldots,l) \tag{206}$$

These relationships express the dependence of l time derivatives of the motion variables, say $\dot{u}_{c+1},\ldots,\dot{u}_p$, on the remaining ones $\dot{u}_1,\ldots,\dot{u}_c$. As in Sec. 2.3.1, it is assumed that this partitioning is such that these equations can in fact be solved for $\dot{u}_{c+1},\ldots,\dot{u}_p$ in terms of $\dot{u}_1,\ldots,\dot{u}_c$. With a relationship for ${}^{N}\mathbf{a}^{P_i}$ in terms of $\dot{u}_1,\ldots,\dot{u}_p$ in hand, one simply embeds the acceleration level constraint equations by expressing $\dot{u}_{c+1},\ldots,\dot{u}_p$ in terms of $\dot{u}_1,\ldots,\dot{u}_c$, and ${}^{N}\tilde{\mathbf{a}}_r^{P_i}$ are subsequently obtained in the same way as partial velocities, namely by inspecting the resulting expression for acceleration to determine the vector coefficients of $\dot{u}_r$ for $r=1,\ldots,c$.

By employing the approach taken in the development of Sec. 2.3.1 it can be shown that the constraint forces constructed with Eqs. (195) make contributions to $\tilde{F}_r$ given by

$$(\tilde{F}_r)_{\mathcal{C}} = \sum_{i=1}^{\nu} {}^{N}\mathbf{a}_r^{P_i}\cdot\sum_{s=1}^{l}\lambda_s\mathbf{W}_{is} = \sum_{s=1}^{l}\lambda_s\alpha_{sr} \qquad (r=1,\ldots,p) \tag{207}$$

where α_{sr} has the same meaning as in Eqs. (204). It can also be demonstrated that the same constraint forces make no contributions to $\tilde{\tilde{F}}_r$, which is to say

$$(\tilde{\tilde{F}}_r)_{\mathcal{C}} = \sum_{i=1}^{\nu} {}^{N}\tilde{\mathbf{a}}_r^{P_i}\cdot\sum_{s=1}^{l}\lambda_s\mathbf{W}_{is} = 0 \qquad (r=1,\ldots,c) \tag{208}$$

and

$$\sum_{i=1}^{\nu} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \mathbf{W}_{is} = 0 \qquad (r = 1, \ldots, c;\ s = 1, \ldots, l) \tag{209}$$

5.1.1 Parallel Velocities

In this subsection and the two that follow, examples provide illustrations of the application of Eqs. (196) and (197) to form equations of motion in which constraint forces respectively are and are not in evidence. Each example involves a system of two individual particles and a nonholonomic constraint equation that is inherently nonlinear. Implementation of the constraint would require the sort of computations that are associated with a control system, as well as ideal actuators and sensors; thus, each example features a servo-constraint. The demonstration is followed in each case by discussion of a similar example from the literature in which the constraint is imposed by purely mechanical means, and it is shown that the nonholonomic constraint equation can in that case be expressed as a linear relationship. The first example concerns a requirement that the velocity in a Newtonian reference frame N of one particle must remain parallel to the velocity in N of another particle.

> **Example 3** Two pucks moving on an air-bearing table fixed in a Newtonian reference frame N are modeled as particles P_1 with a mass of m_1, and P_2 with a mass of m_2. Let two orthogonal unit vectors $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ be fixed in N and define the plane of the table, and let unit vector $\hat{\mathbf{n}}_3 \triangleq \hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2$ be normal to the plane. An external force $\mathbf{f}_1 = \sigma_1\hat{\mathbf{n}}_1 + \sigma_2\hat{\mathbf{n}}_2$ is applied to P_1 whereas a force $\mathbf{f}_2 = \sigma_3\hat{\mathbf{n}}_1 + \sigma_4\hat{\mathbf{n}}_2$ is applied to P_2. The motion of this system is regarded as unconstrained. Suppose that the velocities ${}^N\mathbf{v}^{P_1}$ and ${}^N\mathbf{v}^{P_2}$ of P_1 and P_2 in N are to be constrained such that they must remain parallel at all times. Find expressions for the constraint forces that must be applied to P_1 and P_2 in order for the constraint to be obeyed. A constraint force can be applied to a puck, for example, by four orthogonally mounted thrusters. Let $m_1 = 1$ kg, $m_2 = 2$

kg, and let $\mathbf{f}_1$ and $\mathbf{f}_2$ be characterized by the constants $\sigma_1 = 0.3$ N, $\sigma_2 = 0$ N, $\sigma_3 = 0.7$ N, and $\sigma_4 = 0$ N. At $t = 0$ the velocities of P_1 and P_2 in N are given by ${}^N\mathbf{v}^{P_1} = 0.25\hat{\mathbf{n}}_1 + 1.0\hat{\mathbf{n}}_2$ m/s, and ${}^N\mathbf{v}^{P_2} = 1.0\hat{\mathbf{n}}_1 + 4.0\hat{\mathbf{n}}_2$ m/s. The initial position vectors $\mathbf{p}_i$ from a point O fixed in N to P_i are given by $\mathbf{p}_1 = 1\hat{\mathbf{n}}_2$ m, and $\mathbf{p}_2 = 1\hat{\mathbf{n}}_1$ m. Calculate the value of the multiplier λ associated with the velocity constraint for $0 \le t \le 4$ sec, and verify that the constraint forces keep ${}^N\mathbf{v}^{P_2}$ parallel to ${}^N\mathbf{v}^{P_1}$ during this interval.

The constraint can be expressed as follows. The vector $\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}$ is perpendicular to $\hat{\mathbf{n}}_3$ and to ${}^N\mathbf{v}^{P_1}$ by construction; therefore, requiring ${}^N\mathbf{v}^{P_2}$ to be parallel to ${}^N\mathbf{v}^{P_1}$ is the same as requiring

$$ {}^N\mathbf{v}^{P_2} \cdot (\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}) = 0 \tag{210} $$

This constraint is nonlinear in the velocity vectors because more than one velocity appears in a dot product; it is also nonlinear in motion variables, as will become apparent. Differentiation with respect to t in N brings the constraint equation to the acceleration level, where it is seen to be linear in the acceleration vectors because only one such vector appears in each dot product.

$$ {}^N\mathbf{a}^{P_2} \cdot (\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}) - {}^N\mathbf{a}^{P_1} \cdot (\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_2}) = 0 \tag{211} $$

With the aid of the results obtained in Sec. 2.1.4, it is seen that the constraint requires application of the forces

$$ \mathbf{C}_2 = \lambda(\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}), \qquad \mathbf{C}_1 = -\lambda(\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_2}) \tag{212} $$

to P_2 and P_1 respectively. The constraint forces $\mathbf{C}_1$ and $\mathbf{C}_2$ need not be of equal magnitudes because the constraint does not require ${}^N\mathbf{v}^{P_2}$ and ${}^N\mathbf{v}^{P_1}$ to be equal in magnitude. Moreover, $\mathbf{C}_1$ and $\mathbf{C}_2$ may have the same direction or opposite directions depending on whether the directions of ${}^N\mathbf{v}^{P_1}$ and ${}^N\mathbf{v}^{P_2}$ are opposite or the same.

The unconstrained system possesses four degrees of freedom in N, thus the motion can be characterized by four motion variables defined operationally as

$$^N\mathbf{v}^{P_1} = u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2, \qquad ^N\mathbf{v}^{P_2} = u_3\hat{\mathbf{n}}_1 + u_4\hat{\mathbf{n}}_2 \tag{213}$$

Dynamical equations of motion formed according to Eqs. (196) are readily written as

$$m_1\dot{u}_1 = \sigma_1 + \lambda u_4, \quad m_1\dot{u}_2 = \sigma_2 - \lambda u_3, \quad m_2\dot{u}_3 = \sigma_3 - \lambda u_2, \quad m_2\dot{u}_4 = \sigma_4 + \lambda u_1 \tag{214}$$

The constraint expressed at the velocity level in vector form by Eq. (210) becomes, in scalar form,

$$u_1u_4 - u_2u_3 = 0 \tag{215}$$

This relationship is nonlinear in the motion variables. The constraint at the acceleration level is, however, linear in the time derivatives of the motion variables,

$$u_4\dot{u}_1 - u_3\dot{u}_2 - u_2\dot{u}_3 + u_1\dot{u}_4 = 0 \tag{216}$$

An analytical solution of the linear system of equations (214) and (216) for the five unknowns is manageable, and is given by

$$\lambda = \frac{m_1(\sigma_3u_2 - \sigma_4u_1) + m_2(\sigma_2u_3 - \sigma_1u_4)}{m_1({u_1}^2 + {u_2}^2) + m_2({u_3}^2 + {u_4}^2)} \tag{217}$$

$$\dot{u}_1 = \frac{\sigma_1 + \lambda u_4}{m_1}, \quad \dot{u}_2 = \frac{\sigma_2 - \lambda u_3}{m_1}, \quad \dot{u}_3 = \frac{\sigma_3 - \lambda u_2}{m_2}, \quad \dot{u}_4 = \frac{\sigma_4 + \lambda u_1}{m_2} \tag{218}$$

The configuration of P_1 and P_2 in N is described by four generalized coordinates introduced operationally as

$$\mathbf{p}_1 = q_1\hat{\mathbf{n}}_1 + q_2\hat{\mathbf{n}}_2, \qquad \mathbf{p}_2 = q_3\hat{\mathbf{n}}_1 + q_4\hat{\mathbf{n}}_2 \tag{219}$$

Four kinematical differential equations are given simply by

$$\dot{q}_r = u_r \qquad (r = 1, 2, 3, 4) \tag{220}$$

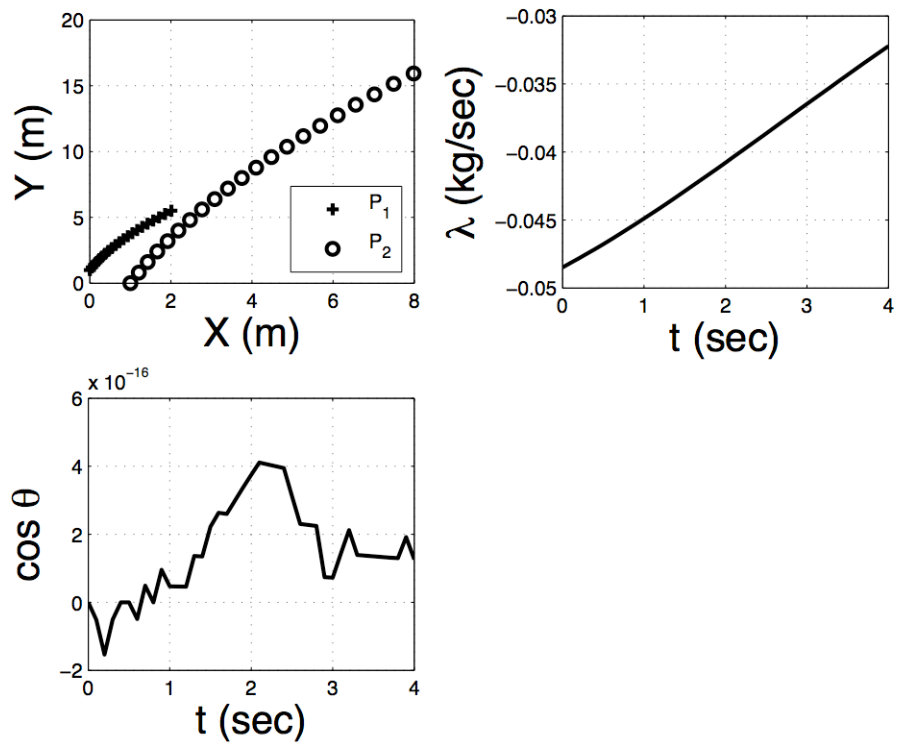


Figure 21: Two Particles with Parallel Velocities

The dynamical and kinematical differential equations are integrated numerically with a variable step-size algorithm, using an absolute error of 1×10^{-8} and a relative error of 1×10^{-7}. Figure 21 contains a plot in the upper left showing the paths of P_1 and P_2, and a time history of λ is plotted in the upper right. The constraint requires ${}^N\mathbf{v}^{P_2}$ to remain perpendicular to $\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}$; therefore, the cosine of the angle between the two vectors calculated as $\cos\theta = {}^N\mathbf{v}^{P_2} \cdot (\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1})/(|{}^N\mathbf{v}^{P_2}||\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}|)$, which should be 0, can be used as a measure of the failure of the numerical solution to satisfy the constraint. The plot in the lower left of Figure 21 shows that the solution meets the constraint extremely well.

One can eliminate completely the small error evident in the time history of $\cos\theta$ and remove λ from the dynamical equations of motion by appealing to

Eqs. (197). Starting with the accelerations in N of P_1 and P_2 expressed as

$$ {}^N\mathbf{a}^{P_1} = \dot{u}_1\hat{\mathbf{n}}_1 + \dot{u}_2\hat{\mathbf{n}}_2, \qquad {}^N\mathbf{a}^{P_2} = \dot{u}_3\hat{\mathbf{n}}_1 + \dot{u}_4\hat{\mathbf{n}}_2 \tag{221} $$

and substituting the expression for $\dot{u}_4$ obtained from Eq. (216), one arrives at

$$ {}^N\mathbf{a}^{P_1} = \dot{u}_1\hat{\mathbf{n}}_1 + \dot{u}_2\hat{\mathbf{n}}_2, \qquad {}^N\mathbf{a}^{P_2} = \dot{u}_3\hat{\mathbf{n}}_1 + \frac{1}{u_1}(-u_4\dot{u}_1 + u_3\dot{u}_2 + u_2\dot{u}_3)\hat{\mathbf{n}}_2 \tag{222} $$

The nonholonomic partial accelerations of P_1 and P_2 in N are identified as

$$ {}^N\tilde{\mathbf{a}}_1^{P_1} = \hat{\mathbf{n}}_1, \qquad {}^N\tilde{\mathbf{a}}_2^{P_1} = \hat{\mathbf{n}}_2, \qquad {}^N\tilde{\mathbf{a}}_3^{P_1} = \mathbf{0} \tag{223} $$

$$ {}^N\tilde{\mathbf{a}}_1^{P_2} = -\frac{u_4}{u_1}\hat{\mathbf{n}}_2, \qquad {}^N\tilde{\mathbf{a}}_2^{P_2} = \frac{u_3}{u_1}\hat{\mathbf{n}}_2, \qquad {}^N\tilde{\mathbf{a}}_3^{P_2} = \hat{\mathbf{n}}_1 + \frac{u_2}{u_1}\hat{\mathbf{n}}_2 \tag{224} $$

With these partial accelerations in hand, nonholonomic generalized active forces are formed according to the expressions

$$ \approx{F}_r = {}^N\tilde{\mathbf{a}}_r^{P_1} \cdot (\mathbf{f}_1 - \lambda\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_2}) + {}^N\tilde{\mathbf{a}}_r^{P_2} \cdot (\mathbf{f}_2 + \lambda\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}) \qquad (r = 1, 2, 3) \tag{225} $$

The first of these is given by

$$ \begin{aligned} \overset{\approx}{F}_1 &= \hat{\mathbf{n}}_1 \cdot (\mathbf{f}_1 - \lambda\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_2}) - \frac{u_4}{u_1}\hat{\mathbf{n}}_2 \cdot (\mathbf{f}_2 + \lambda\hat{\mathbf{n}}_3 \times {}^N\mathbf{v}^{P_1}) \\ &= \sigma_1 + \lambda u_4 - \frac{u_4}{u_1}(\sigma_4 + \lambda u_1) \\ &= \sigma_1 - \frac{u_4}{u_1}\sigma_4 \end{aligned} \tag{226} $$

Similarly,

$$ \overset{\approx}{F}_2 = \sigma_2 + \frac{u_3}{u_1}\sigma_4 \tag{227} $$

$$ \overset{\approx}{F}_3 = \sigma_3 + \frac{u_2}{u_1}\sigma_4 \tag{228} $$

The multiplier λ is clearly eliminated from $\overset{\approx}{F}_1$, $\overset{\approx}{F}_2$, and $\overset{\approx}{F}_3$, and thus the constraint forces $\mathbf{C}_1$ and $\mathbf{C}_2$ do not contribute to the reduced equations of motion. Nonholonomic generalized inertia forces are given by

$$ \overset{\approx}{F}{}_r^{\star} = {}^N\tilde{\mathbf{a}}_r^{P_1} \cdot (-m_1\, {}^N\mathbf{a}^{P_1}) + {}^N\tilde{\mathbf{a}}_r^{P_2} \cdot (-m_2\, {}^N\mathbf{a}^{P_2}) \qquad (r = 1, 2, 3) \tag{229} $$

or

$$\begin{aligned}\tilde{\tilde{F}}_1^\star &= -m_1\dot{u}_1 + m_2\frac{u_4}{{u_1}^2}(-u_4\dot{u}_1 + u_3\dot{u}_2 + u_2\dot{u}_3) \\ &= -\left[m_1 + m_2\left(\frac{u_4}{u_1}\right)^2\right]\dot{u}_1 + m_2\frac{u_3u_4}{{u_1}^2}\dot{u}_2 + m_2\frac{u_2u_4}{{u_1}^2}\dot{u}_3\end{aligned} \tag{230}$$

$$\begin{aligned}\tilde{\tilde{F}}_2^\star &= -m_1\dot{u}_2 - m_2\frac{u_3}{{u_1}^2}(-u_4\dot{u}_1 + u_3\dot{u}_2 + u_2\dot{u}_3) \\ &= m_2\frac{u_3u_4}{{u_1}^2}\dot{u}_1 - \left[m_1 + m_2\left(\frac{u_3}{u_1}\right)^2\right]\dot{u}_2 - m_2\frac{u_2u_3}{{u_1}^2}\dot{u}_3\end{aligned} \tag{231}$$

$$\begin{aligned}\tilde{\tilde{F}}_3^\star &= -m_2\dot{u}_3 - m_2\frac{u_2}{{u_1}^2}(-u_4\dot{u}_1 + u_3\dot{u}_2 + u_2\dot{u}_3) \\ &= m_2\frac{u_2u_4}{{u_1}^2}\dot{u}_1 - m_2\frac{u_2u_3}{{u_1}^2}\dot{u}_2 - m_2\left[1 + \left(\frac{u_2}{u_1}\right)^2\right]\dot{u}_3\end{aligned} \tag{232}$$

The mass matrix associated with these equations of motion is symmetric. After expressing u_4 as u_2u_3/u_1 as required by Eq. (215), the reduced dynamical equations of motion $\tilde{\tilde{F}}_r + \tilde{\tilde{F}}_r^\star = 0$ $(r = 1, 2, 3)$ and the kinematical differential equations (220) are integrated numerically using the initial conditions given in the problem statement. The paths of P_1 and P_2 are indistinguishable from those shown in Figure 21, and $\cos\theta$ is identically 0 throughout the simulation.

The first example in Refs. [85] and [86] is similar to the preceding situation, but an additional configuration constraint is imposed on P_1 and P_2; they are connected by a rod of fixed length $2L$. It is said that the requirement of parallel velocities can be achieved in practice by attaching at the rod's midpoint a sharp-edged circular disk, or blade, that is perpendicular to the rod. A relationship having the form of Eq. (215) is given, and put forth as an example of a nonlinear nonholonomic constraint equation. However, in this instance the constraint dictated by the blade can in fact be described by a linear nonholonomic constraint equation. The configuration in N of the rigid body B formed by the rod, particles, and blade can be specified with three generalized coordinates; q_1 and q_2 to give the position of, say, the rod's midpoint

$B^\star$, and one angle q_3 to specify the orientation of the rod. Three motion variables characterizing the motion of B in N may be introduced through the operational definitions ${}^N\mathbf{v}^{B^\star} = u_1\hat{\mathbf{b}}_1 + u_2\hat{\mathbf{b}}_2$ and ${}^N\boldsymbol{\omega}^B = u_3\hat{\mathbf{n}}_3$, where perpendicular unit vectors $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$ are fixed in B and lie in the plane of motion with $\hat{\mathbf{b}}_1$ parallel to the rod. The blade prevents ${}^N\mathbf{v}^{B^\star}$ from having a component in the direction of $\hat{\mathbf{b}}_1$; therefore the constraint can be stated as ${}^N\mathbf{v}^{B^\star} \cdot \hat{\mathbf{b}}_1 = u_1 = 0$, which is clearly linear both in the vector ${}^N\mathbf{v}^{B^\star}$ and in the motion variable u_1. There appears to be some recognition of this in Ref. [85]. The directions of the constraint forces obtained in Eqs. (212) are seen to be the same as those indicated in the diagram on the right side of Fig. 2a in Ref. [85].

5.1.2 Velocities of Equal Magnitude

A second illustration of the use of Eqs. (196) and (197) is provided by requiring the inertial velocities of two particles to have equal magnitudes. The equations of motion produced in each case are similar to those obtained in Example 3 (see Sec. 5.1.1).

> **Example 4** Suppose that the velocities in N of the two pucks in Example 3 are required to have equal magnitudes rather than parallel directions. Find expressions for the constraint forces that must be applied to P_1 and P_2 in order to ensure adherence to the relationships of constraint. Let $m_1 = 1$ kg, $m_2 = 2$ kg, and let $\mathbf{f}_1$ and $\mathbf{f}_2$ be characterized by the constants $\sigma_1 = 1.0$ N, $\sigma_2 = 0$ N, $\sigma_3 = 1.0$ N, and $\sigma_4 = 0$ N. At $t = 0$ the velocities of P_1 and P_2 in N are given by ${}^N\mathbf{v}^{P_1} = 0.25\hat{\mathbf{n}}_1 + 1.0\hat{\mathbf{n}}_2$ m/s, and ${}^N\mathbf{v}^{P_2} = 0.25\hat{\mathbf{n}}_1 - 1.0\hat{\mathbf{n}}_2$ m/s. The initial position vectors $\mathbf{p}_i$ from a point O fixed in N to P_i are given by $\mathbf{p}_1 = 2\hat{\mathbf{n}}_1 + 1\hat{\mathbf{n}}_2$ m, and $\mathbf{p}_2 = 1\hat{\mathbf{n}}_1$ m. Calculate the value of the multiplier λ associated with the velocity constraint for $0 \le t \le 4$ sec, and verify that the constraint forces keep the magnitude of ${}^N\mathbf{v}^{P_2}$ equal to ${}^N\mathbf{v}^{P_1}$ during this interval.

The constraint can be expressed by the relationship

$$ {}^N\mathbf{v}^{P_2} \cdot {}^N\mathbf{v}^{P_2} - {}^N\mathbf{v}^{P_1} \cdot {}^N\mathbf{v}^{P_1} = 0 \tag{233} $$

which is nonlinear in the velocity vectors and, as seen momentarily, in motion variables. The acceleration level of the constraint equation is linear in the acceleration vectors,

$$ {}^N\mathbf{a}^{P_2} \cdot {}^N\mathbf{v}^{P_2} - {}^N\mathbf{a}^{P_1} \cdot {}^N\mathbf{v}^{P_1} = 0 \tag{234} $$

According to Sec. 2.1.4, the constraint requires application of the forces

$$ \mathbf{C}_2 = \lambda\, {}^N\mathbf{v}^{P_2}, \qquad \mathbf{C}_1 = -\lambda\, {}^N\mathbf{v}^{P_1} \tag{235} $$

to P_2 and P_1 respectively. It is seen that $\mathbf{C}_1$ and $\mathbf{C}_2$ have equal magnitudes when the constraint is obeyed.

With four motion variables having the same meanings as in Eqs. (213), dynamical equations of motion formed with reference to Eqs. (196) are given by

$$ m_1\dot{u}_1 = \sigma_1 - \lambda u_1, \quad m_1\dot{u}_2 = \sigma_2 - \lambda u_2, \quad m_2\dot{u}_3 = \sigma_3 + \lambda u_3, \quad m_2\dot{u}_4 = \sigma_4 + \lambda u_4 \tag{236} $$

In scalar form, Eq. (233) is nonlinear in the motion variables,

$$ {u_3}^2 + {u_4}^2 - ({u_1}^2 + {u_2}^2) = 0 \tag{237} $$

whereas Eq. (234) is linear in the time derivatives of the motion variables,

$$ -u_1\dot{u}_1 - u_2\dot{u}_2 + u_3\dot{u}_3 + u_4\dot{u}_4 = 0 \tag{238} $$

An analytical solution of the linear system of equations (236) and (238) for the five unknowns yields

$$ \lambda = \frac{m_2(\sigma_1 u_1 + \sigma_2 u_2) - m_1(\sigma_3 u_3 + \sigma_4 u_4)}{m_2({u_1}^2 + {u_2}^2) + m_1({u_3}^2 + {u_4}^2)} \tag{239} $$

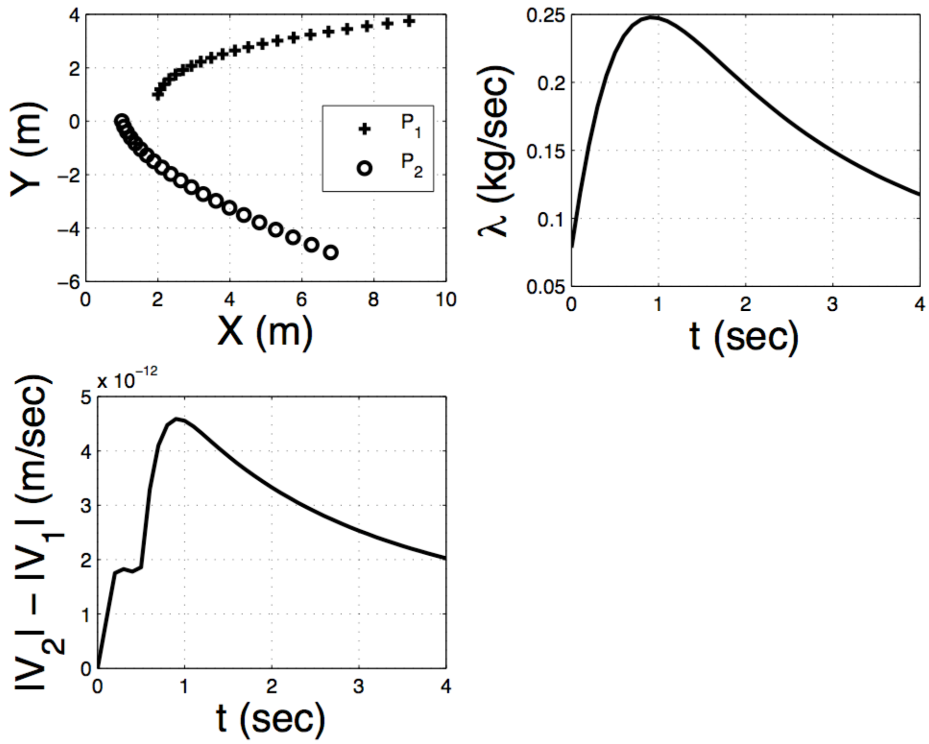


Figure 22: Two Particles with Velocities Equal in Magnitude

$$\dot{u}_1 = \frac{\sigma_1 - \lambda u_1}{m_1}, \quad \dot{u}_2 = \frac{\sigma_2 - \lambda u_2}{m_1}, \quad \dot{u}_3 = \frac{\sigma_3 + \lambda u_3}{m_2}, \quad \dot{u}_4 = \frac{\sigma_4 + \lambda u_4}{m_2} \tag{240}$$

Four generalized coordinates that describe the configuration of P_1 and P_2 in N are once again introduced as in Eqs. (219), and Eqs. (220) provide the four associated kinematical differential equations.

As in Example 3, a variable step-size algorithm is used in numerical integration of the dynamical and kinematical differential equations, using an absolute error of 1×10^{-8} and a relative error of 1×10^{-7}. The trajectories of P_1 and P_2, and a time history of λ are shown in Figure 22. The constraint requires $|{}^N\mathbf{v}^{P_2}| = |{}^N\mathbf{v}^{P_1}|$, therefore the difference $|{}^N\mathbf{v}^{P_2}| - |{}^N\mathbf{v}^{P_1}|$ plotted in the lower left of Figure 22 indicates how well the numerical solution fulfills the condition of constraint.

If the constraint forces are not of interest one may work with a minimal set of dynamical equations obtained by carrying out the operations indicated in Eqs. (197). Substitution of the expression for $\dot{u}_4$ obtained from Eq. (238) into Eqs. (221) yields

$$^N\mathbf{a}^{P_1} = \dot{u}_1\hat{\mathbf{n}}_1 + \dot{u}_2\hat{\mathbf{n}}_2, \qquad ^N\mathbf{a}^{P_2} = \dot{u}_3\hat{\mathbf{n}}_1 + \frac{1}{u_4}(u_1\dot{u}_1 + u_2\dot{u}_2 - u_3\dot{u}_3)\hat{\mathbf{n}}_2 \tag{241}$$

from which the nonholonomic partial accelerations of P_1 and P_2 in N are identified by inspection.

$$^N\tilde{\mathbf{a}}_1^{P_1} = \hat{\mathbf{n}}_1, \qquad ^N\tilde{\mathbf{a}}_2^{P_1} = \hat{\mathbf{n}}_2, \qquad ^N\tilde{\mathbf{a}}_3^{P_1} = \mathbf{0} \tag{242}$$

$$^N\tilde{\mathbf{a}}_1^{P_2} = \frac{u_1}{u_4}\hat{\mathbf{n}}_2, \qquad ^N\tilde{\mathbf{a}}_2^{P_2} = \frac{u_2}{u_4}\hat{\mathbf{n}}_2, \qquad ^N\tilde{\mathbf{a}}_3^{P_2} = \hat{\mathbf{n}}_1 - \frac{u_3}{u_4}\hat{\mathbf{n}}_2 \tag{243}$$

The next step is to form nonholonomic generalized active forces,

$$\approx{F}_r = {}^N\tilde{\mathbf{a}}_r^{P_1} \cdot (\mathbf{f}_1 - \lambda\, ^N\mathbf{v}^{P_1}) + {}^N\tilde{\mathbf{a}}_r^{P_2} \cdot (\mathbf{f}_2 + \lambda\, ^N\mathbf{v}^{P_2}) \qquad (r = 1, 2, 3) \tag{244}$$

or

$$\overset{\approx}{F}_1 = \sigma_1 + \frac{u_1}{u_4}\sigma_4, \qquad \overset{\approx}{F}_2 = \sigma_2 + \frac{u_2}{u_4}\sigma_4, \qquad \overset{\approx}{F}_3 = \sigma_3 - \frac{u_3}{u_4}\sigma_4 \tag{245}$$

Absence of evidence of λ shows that the constraint forces $\mathbf{C}_1$ and $\mathbf{C}_2$ are noncontributing to $\overset{\approx}{F}_1$, $\overset{\approx}{F}_2$, and $\overset{\approx}{F}_3$. The final step to be carried out is the formation of nonholonomic generalized inertia forces as indicated in Eqs. (229).

$$\overset{\approx}{F}{}_1^\star = -\left[m_1 + m_2\left(\frac{u_1}{u_4}\right)^2\right]\dot{u}_1 - m_2\frac{u_1u_2}{{u_4}^2}\dot{u}_2 + m_2\frac{u_1u_3}{{u_4}^2}\dot{u}_3 \tag{246}$$

$$\overset{\approx}{F}{}_2^\star = -m_2\frac{u_1u_2}{{u_4}^2}\dot{u}_1 - \left[m_1 + m_2\left(\frac{u_2}{u_4}\right)^2\right]\dot{u}_2 + m_2\frac{u_2u_3}{{u_4}^2}\dot{u}_3 \tag{247}$$

$$\overset{\approx}{F}{}_3^\star = m_2\frac{u_1u_3}{{u_4}^2}\dot{u}_1 + m_2\frac{u_2u_3}{{u_4}^2}\dot{u}_2 - m_2\left[1 + \left(\frac{u_3}{u_4}\right)^2\right]\dot{u}_3 \tag{248}$$

The mass matrix is once again observed to be symmetric. After expressing u_4 as $-\sqrt{{u_1}^2 + {u_2}^2 - {u_3}^2}$ in accordance with Eq. (237), the reduced dynamical equations of motion $\overset{\approx}{F}_r + \overset{\approx}{F}{}_r^\star = 0$ $(r = 1, 2, 3)$ and the kinematical differential equations (220) are integrated numerically using the initial conditions given in the problem statement. The paths of P_1 and P_2 are identical to those shown in Figure 22, and the absolute value of $|{}^N\mathbf{v}^{P_2}| - |{}^N\mathbf{v}^{P_1}|$ remains less than 2.22×10^{-16} throughout the simulation.

The second example in Ref. [85] involves two particles whose velocities are to remain equal in magnitude; however, an additional configuration constraint is imposed on P_1 and P_2 as they are connected by a rod of fixed length. Zekovich observes the velocities are made equal in magnitude by placing a blade at the rod's midpoint and making the edge parallel to the rod. An expression with the same form as Eq. (237) is offered as a nonlinear nonholonomic constraint equation. As is the case with Zekovich's first example, it can easily be shown that a linear nonholonomic constraint equation describes the constraint dictated by the blade. Once again, let B denote the rigid body formed by the rod, particles, and blade, and let $B^\star$ be the rod's midpoint. Designate three motion variables to be such that ${}^N\mathbf{v}^{B^\star} = u_1\hat{\mathbf{b}}_1 + u_2\hat{\mathbf{b}}_2$ and ${}^N\boldsymbol{\omega}^B = u_3\hat{\mathbf{n}}_3$, where perpendicular unit vectors $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$ are fixed in B and lie in the plane of motion with $\hat{\mathbf{b}}_1$ parallel to the rod. In this case the blade prevents ${}^N\mathbf{v}^{B^\star}$ from having a component in the direction of $\hat{\mathbf{b}}_2$ and the constraint is stated simply as ${}^N\mathbf{v}^{B^\star} \cdot \hat{\mathbf{b}}_2 = u_2 = 0$, which is clearly linear both in the vector ${}^N\mathbf{v}^{B^\star}$ and in the motion variable u_2. The diagram on the right side of Fig. 2b in Ref. [85] shows a constraint force in the direction of ${}^N\mathbf{v}^{P_1}$ and the other constraint force in the direction opposite to ${}^N\mathbf{v}^{P_2}$; this result can be made to agree with Eqs. (235) by renaming the two particles.

5.1.3 Perpendicular Velocities

A third demonstration of employing Eqs. (196) and (197) to obtain equations of motion can be given by imposing a constraint on the orientation of the inertial velocities of P_1 and P_2. In this case it is required that the velocities remain orthogonal.

Example 5 Require the velocities in N of the two pucks in the Examples 3 and 4 (see Secs. 5.1.1 and 5.1.2) to have perpendicular directions. Find expressions for the constraint forces that must be applied to P_1 and P_2 in order for the constraint to be obeyed. Let $m_1 = 1$ kg, $m_2 = 2$ kg, and let $\mathbf{f}_1$ and $\mathbf{f}_2$ be characterized by the constants $\sigma_1 = 1.0$ N, $\sigma_2 = 0$ N, $\sigma_3 = 1.0$ N, and $\sigma_4 = 0$ N. At $t = 0$ the velocities of P_1 and P_2 in N are given by ${}^N\mathbf{v}^{P_1} = 0.3\hat{\mathbf{n}}_1 + 0.4\hat{\mathbf{n}}_2$ m/s, and ${}^N\mathbf{v}^{P_2} = 0.4\hat{\mathbf{n}}_1 - 0.3\hat{\mathbf{n}}_2$ m/s. The initial position vectors $\mathbf{p}_i$ from a point O fixed in N to P_i are given by $\mathbf{p}_1 = 1\hat{\mathbf{n}}_1 - 2\hat{\mathbf{n}}_2$ m, and $\mathbf{p}_2 = 1\hat{\mathbf{n}}_1 + 2\hat{\mathbf{n}}_2$ m. Calculate the value of the multiplier λ associated with the constraint for $0 \leq t \leq 4$ sec, and verify that the constraint forces keep ${}^N\mathbf{v}^{P_2}$ orthogonal to ${}^N\mathbf{v}^{P_1}$ during this interval.

The constraint can be expressed by the nonlinear velocity relationship

$$ {}^N\mathbf{v}^{P_2} \cdot {}^N\mathbf{v}^{P_1} = 0 \tag{249} $$

At the acceleration level the constraint equation is written as

$$ {}^N\mathbf{a}^{P_2} \cdot {}^N\mathbf{v}^{P_1} + {}^N\mathbf{a}^{P_1} \cdot {}^N\mathbf{v}^{P_2} = 0 \tag{250} $$

In view of Sec. 2.1.4, the constraint requires application of the forces

$$ \mathbf{C}_2 = \lambda\, {}^N\mathbf{v}^{P_1}, \qquad \mathbf{C}_1 = \lambda\, {}^N\mathbf{v}^{P_2} \tag{251} $$

to P_2 and P_1 respectively. The constraint forces $\mathbf{C}_1$ and $\mathbf{C}_2$ need not have equal magnitudes; $\mathbf{C}_1$ is perpendicular to $\mathbf{C}_2$ when the constraint is satisfied.

Dynamical equations of motion constructed in accordance with Eqs. (196) are given by

$$ m_1\dot{u}_1 = \sigma_1 + \lambda u_3, \quad m_1\dot{u}_2 = \sigma_2 + \lambda u_4, \quad m_2\dot{u}_3 = \sigma_3 + \lambda u_1, \quad m_2\dot{u}_4 = \sigma_4 + \lambda u_2 \tag{252} $$

where the motion variables u_1, u_2, u_3, and u_4 are defined in Eqs. (213). When stated in scalar form, Eq. (249) is nonlinear in the motion variables,

$$u_1 u_3 + u_2 u_4 = 0 \tag{253}$$

and Eq. (250) is linear in the time derivatives of the motion variables,

$$u_3 \dot{u}_1 + u_4 \dot{u}_2 + u_1 \dot{u}_3 + u_2 \dot{u}_4 = 0 \tag{254}$$

Equations (252) and (254) can be solved for the five unknowns to obtain

$$\lambda = -\frac{m_1(\sigma_3 u_1 + \sigma_4 u_2) + m_2(\sigma_1 u_3 + \sigma_2 u_4)}{m_1({u_1}^2 + {u_2}^2) + m_2({u_3}^2 + {u_4}^2)} \tag{255}$$

$$\dot{u}_1 = \frac{\sigma_1 + \lambda u_3}{m_1}, \quad \dot{u}_2 = \frac{\sigma_2 + \lambda u_4}{m_1}, \quad \dot{u}_3 = \frac{\sigma_3 + \lambda u_1}{m_2}, \quad \dot{u}_4 = \frac{\sigma_4 + \lambda u_2}{m_2} \tag{256}$$

The four generalized coordinates introduced in Eqs. (219) and the associated kinematical differential equations (220) are employed once more.

As in Examples 3 and 4, numerical solution of the dynamical and kinematical differential equations is obtained with a variable step-size algorithm, using an absolute error of 1×10^{-8} and a relative error of 1×10^{-7}. The unconstrained trajectories ($\lambda = 0$) of P_1 and P_2 are displayed in the upper left of Figure 23, to be compared to the constrained trajectories shown in the upper right. It is clear that ${}^N\mathbf{v}^{P_1}$ and ${}^N\mathbf{v}^{P_2}$ are becoming parallel in the absence of constraint forces, whereas they remain perpendicular when $\mathbf{C}_1$ and $\mathbf{C}_2$ are applied. A time history of λ is shown in the lower left of Figure 23. The constraint requires ${}^N\mathbf{v}^{P_2}$ to remain perpendicular to ${}^N\mathbf{v}^{P_1}$; hence, the cosine of the angle between the two vectors calculated as $\cos\theta = {}^N\mathbf{v}^{P_2} \cdot {}^N\mathbf{v}^{P_1} / (|{}^N\mathbf{v}^{P_2}||{}^N\mathbf{v}^{P_1}|)$ quantifies the constraint violation in the numerical solution. As seen in the lower right of Figure 23, orthogonality is preserved very well.

A minimal set of dynamical equations can be obtained after identifying the nonholonomic partial accelerations of P_1 and P_2 in N by inspection as usual.

$${}^N\tilde{\mathbf{a}}_1^{P_1} = \hat{\mathbf{n}}_1, \qquad {}^N\tilde{\mathbf{a}}_2^{P_1} = \hat{\mathbf{n}}_2, \qquad {}^N\tilde{\mathbf{a}}_3^{P_1} = \mathbf{0} \tag{257}$$

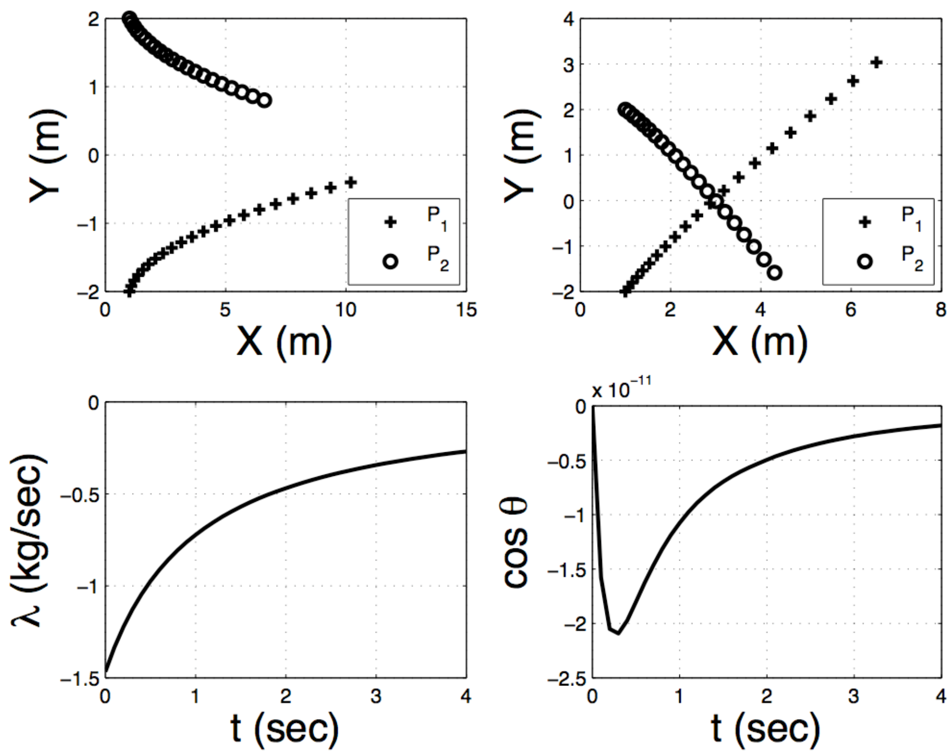


Figure 23: Two Particles with Perpendicular Velocities

$$
{}^N\tilde{\mathbf{a}}_1^{P_2} = -\frac{u_3}{u_2}\hat{\mathbf{n}}_2, \qquad {}^N\tilde{\mathbf{a}}_2^{P_2} = -\frac{u_4}{u_2}\hat{\mathbf{n}}_2, \qquad {}^N\tilde{\mathbf{a}}_3^{P_2} = \hat{\mathbf{n}}_1 - \frac{u_1}{u_2}\hat{\mathbf{n}}_2 \tag{258}
$$

The corresponding nonholonomic generalized active forces are as follows,

$$
\overset{\approx}{F}_1 = \sigma_1 - \frac{u_3}{u_2}\sigma_4, \qquad \overset{\approx}{F}_2 = \sigma_2 - \frac{u_4}{u_2}\sigma_4, \qquad \overset{\approx}{F}_3 = \sigma_3 - \frac{u_1}{u_2}\sigma_4 \tag{259}
$$

and the nonholonomic generalized inertia forces are given by

$$
\overset{\approx}{F}{}_1^{\star} = -\left[m_1 + m_2\left(\frac{u_3}{u_2}\right)^2\right]\dot{u}_1 - m_2\frac{u_3u_4}{{u_2}^2}\dot{u}_2 - m_2\frac{u_1u_3}{{u_2}^2}\dot{u}_3 \tag{260}
$$

$$
\overset{\approx}{F}{}_2^{\star} = -m_2\frac{u_3u_4}{{u_2}^2}\dot{u}_1 - \left[m_1 + m_2\left(\frac{u_4}{u_2}\right)^2\right]\dot{u}_2 - m_2\frac{u_1u_4}{{u_2}^2}\dot{u}_3 \tag{261}
$$

$$
\overset{\approx}{F}{}_3^{\star} = -m_2\frac{u_1u_3}{{u_2}^2}\dot{u}_1 - m_2\frac{u_1u_4}{{u_2}^2}\dot{u}_2 - m_2\left[1 + \left(\frac{u_1}{u_2}\right)^2\right]\dot{u}_3 \tag{262}
$$

As is the case in Examples 3 and 4, the mass matrix possesses the property of symmetry. After expressing u_4 as $-u_1u_3/u_2$ in accordance with Eq. (253), the reduced dynamical equations of motion $\overset{\approx}{F}_r + \overset{\approx}{F}{}_r^{\star} = 0$ $(r = 1, 2, 3)$ and the kinematical differential equations (220) are integrated numerically using the initial conditions given in the problem statement. The paths of P_1 and P_2 are identical to those shown in the upper right plot of Figure 23, and the absolute value of $\cos\theta$ remains less than 7.64×10^{-17} throughout the simulation.

In Refs. [85] and [86] Zekovich provides examples in which velocities of two particles are to remain perpendicular to one another. Instead of a rigid rod, P_1 and P_2 are connected by a "fork" that allows relative translation along the line joining P_1 and P_2. In other words, P_1 is regarded as fixed in a rigid body B, and a prismatic joint makes it possible for P_2 to move on B. The development in Ref. [85] is greatly simplified by working with a set of motion variables to be defined presently; furthermore, they are used to show that the relevant nonholonomic constraint equations can be written as linear expressions.

Let perpendicular unit vectors $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$ be fixed in B such that they lie in the plane of motion of P_1 and P_2, and $\hat{\mathbf{b}}_1$ is in the direction of the prismatic joint that permits P_2 to slide on B. Unit vector $\hat{\mathbf{b}}_3$ is perpendicular to $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$, and to the plane of the motion. Four motion variables are introduced operationally by writing ${}^N\mathbf{v}^{P_1} = u_1\hat{\mathbf{b}}_1 + u_2\hat{\mathbf{b}}_2$, ${}^N\boldsymbol{\omega}^B = u_3\hat{\mathbf{b}}_3$, and ${}^B\mathbf{v}^{P_2} = u_4\hat{\mathbf{b}}_1$. Hence, ${}^N\mathbf{v}^{P_2} = (u_1 + u_4)\hat{\mathbf{b}}_1 + (u_2 + q_4u_3)\hat{\mathbf{b}}_2$, where q_4 is the distance between P_1 and P_2. The perpendicular velocity constraint is expressed as ${}^N\mathbf{v}^{P_2} \cdot {}^N\mathbf{v}^{P_1} = u_1(u_1 + u_4) + u_2(u_2 + q_4u_3) = 0$.

Zekovich begins the analysis by attaching a blade at P_1 with the edge perpendicular to $\hat{\mathbf{b}}_1$; the resulting constraint is expressed linearly as ${}^N\mathbf{v}^{P_1} \cdot \hat{\mathbf{b}}_1 = u_1 = 0$, and the corresponding Eq. (8) in Ref. [85] is likewise linear. With $u_1 = 0$, the velocity constraint is rewritten as ${}^N\mathbf{v}^{P_2} \cdot {}^N\mathbf{v}^{P_1} = u_2(u_2 + q_4u_3) = 0$, which corresponds to Eq. (9) of Ref. [85]. Zekovich then notes the constraint can be satisfied in either of two ways. The first possibility is imposition of the constraint expressed by the linear equation ${}^N\mathbf{v}^{P_1} \cdot \hat{\mathbf{b}}_2 = u_2 = 0$, in which case P_1 is fixed in N and the blade at P_1 is no longer necessary. The second possibility also involves a constraint described by a linear relationship ${}^N\mathbf{v}^{P_2} \cdot \hat{\mathbf{b}}_2 = u_2 + q_4 u_3 = 0$; such a restriction can be imposed by fixing a blade at P_2 with the edge orthogonal to $\hat{\mathbf{b}}_2$. The presence of perpendicular constraint forces exerted by perpendicular blades is in keeping with the result of Eqs. (251), although it contradicts the direction of $\mathbf{R}_2$ indicated in Fig. 3a of Ref. [85].

Jankowski has developed an approach for dealing with constraint equations that are not necessarily linear in acceleration. A procedure is set forth in Ref. [41] for forming dynamical equations of motion in which Lagrange multipliers do appear, and then the multipliers are eliminated by employing an orthogonal complement matrix to obtain a reduced set of equations. As mentioned previously, the paper concludes with an example involving a single particle P. It is readily demonstrated that Eqs. (196) and (197) can be used to obtain the results reported in Ref. [41] when the magnitude of the velocity ${}^N\mathbf{v}^P$ of P in N must have a prescribed time history; that is, ${}^N\mathbf{v}^P \cdot {}^N\mathbf{v}^P - v(t)^2 = 0$. Moreover, inspection of this constraint equation at the acceleration level indicates the constraint force applied to P is in the direction of ${}^N\mathbf{v}^P$, and Jankowski reaches the same conclusion. However, Eqs. (196) and (197) are not applicable to the subsequent example in which the magnitude of the acceleration ${}^N\mathbf{a}^P$ of P in N is a prescribed function of the time t, ${}^N\mathbf{a}^P \cdot {}^N\mathbf{a}^P - a(t)^2 = 0$

5.1.4 Appell's Particle

As mentioned previously, the literature contains ample discussion of an example proposed by Appell in which a single particle moving in a uniform gravitational field is subject to a nonlinear nonholonomic constraint equation. In connection with this example, a final brief demonstration of the use of Eqs. (196) and (197) shows that they lead to results obtained in Refs. [69] and [76].

Three motion variables u_1, u_2, and u_3 are introduced such that the velocity ${}^N\mathbf{v}^P$ in a Newtonian reference frame N of a particle P is written as

$$ {}^N\mathbf{v}^P = u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2 + u_3\hat{\mathbf{n}}_3 \tag{263} $$

where $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$, and $\hat{\mathbf{n}}_3$ are a right-handed set of mutually perpendicular unit vectors fixed in N. Appell's restriction on the velocity of P is often expressed by the relationship

$$u_3{}^2 = a^2(u_1{}^2 + u_2{}^2) \tag{264}$$

where a is a constant. It is pointed out in Ref. [69] that the relationship describes a requirement for the angle γ between ${}^N\mathbf{v}^P$ and $\hat{\mathbf{n}}_3$, the vertical direction, to remain constant. In fact, the constant a is $\cos\gamma/\sin\gamma$. The nonlinear nonholonomic constraint equation is differentiated with respect to time to bring it to the acceleration level

$$2u_3\dot{u}_3 = 2a^2(u_1\dot{u}_1 + u_2\dot{u}_2) \tag{265}$$

where it is linear in $\dot{u}_1$, $\dot{u}_2$, and $\dot{u}_3$; it can be rewritten as

$${}^N\mathbf{a}^P \cdot \hat{\mathbf{n}}_3 - \frac{a^2}{u_3}(u_1\,{}^N\mathbf{a}^P \cdot \hat{\mathbf{n}}_1 + u_2\,{}^N\mathbf{a}^P \cdot \hat{\mathbf{n}}_2) = {}^N\mathbf{a}^P \cdot \left[\hat{\mathbf{n}}_3 - \frac{a^2}{u_3}(u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2)\right] = 0 \tag{266}$$

where ${}^N\mathbf{a}^P$ is the acceleration of P in N. Inspection of this equation according to the procedure given in Sec. 2.1.4 indicates that a constraint force $\mathbf{C}$ must be applied to P such that the force is parallel to the vector within the square brackets; that is,

$$\mathbf{C} = \lambda\left[\hat{\mathbf{n}}_3 - \frac{a^2}{u_3}(u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2)\right] \tag{267}$$

This result is in agreement with what is presented in Ref. [69], where it is shown that $\mathbf{C} \cdot {}^N\mathbf{v}^P = 0$ when ${}^N\mathbf{v}^P$ obeys the constraint.

The gravitational force acting on P is denoted by $\mathbf{f} = -mg\hat{\mathbf{n}}_3$ where m is the mass of P and the constant g represents the gravitational force per unit mass. Three dynamical equations of motion obtained with Eqs. (196) can be written in terms of vectors as $\hat{\mathbf{n}}_r \cdot (\mathbf{f} + \mathbf{C} - m\,{}^N\mathbf{a}^P) = 0$ $(r = 1, 2, 3)$, or in terms of scalars

$$m\dot{u}_1 = -\lambda a^2 u_1/u_3, \qquad m\dot{u}_2 = -\lambda a^2 u_2/u_3, \qquad m\dot{u}_3 = \lambda - mg \tag{268}$$

in which case they resemble certain expressions found in Ref. [69]. When one substitutes u_3 obtained from the constraint equation (264), the results are identical to Eqs. (3.7) of Ref. [76],

$$m\dot{u}_1 = -\lambda a \frac{u_1}{\sqrt{{u_1}^2 + {u_2}^2}}, \qquad m\dot{u}_2 = -\lambda a \frac{u_2}{\sqrt{{u_1}^2 + {u_2}^2}}, \qquad m\dot{u}_3 = \lambda - mg \tag{269}$$

The fourth relationship needed to determine the unknowns $\dot{u}_1$, $\dot{u}_2$, $\dot{u}_3$, and λ is provided by Eq. (265); when it is solved for $\dot{u}_3$ and substitution is performed in the third of Eqs. (269), one obtains

$$\lambda = mg + m\frac{a^2}{u_3}(u_1\dot{u}_1 + u_2\dot{u}_2) = mg - \frac{a}{\sqrt{{u_1}^2 + {u_2}^2}}\left[\frac{\lambda a({u_1}^2 + {u_2}^2)}{\sqrt{{u_1}^2 + {u_2}^2}}\right] = mg - \lambda a^2 \tag{270}$$

where the second step is made with the aid of Eq. (264) together with the first and second of Eqs. (269). A solution for λ is now at hand, and it can be used as a replacement in the first and second of Eqs. (269) to yield

$$\lambda = \frac{mg}{1 + a^2} = mg\sin^2\gamma \tag{271}$$

$$\dot{u}_1 = -\frac{gau_1}{(1 + a^2)\sqrt{{u_1}^2 + {u_2}^2}} = -\frac{g\sin\gamma\cos\gamma\, u_1}{\sqrt{{u_1}^2 + {u_2}^2}} \tag{272}$$

$$\dot{u}_2 = -\frac{gau_2}{(1 + a^2)\sqrt{{u_1}^2 + {u_2}^2}} = -\frac{g\sin\gamma\cos\gamma\, u_2}{\sqrt{{u_1}^2 + {u_2}^2}} \tag{273}$$

The dynamical equations of motion (272) and (273) from which λ has been eliminated can be obtained directly, instead, by resorting to Eqs. (197). After embedding the acceleration level constraint equation in ${}^N\mathbf{a}^P$,

$${}^N\mathbf{a}^P = \dot{u}_1\hat{\mathbf{n}}_1 + \dot{u}_2\hat{\mathbf{n}}_2 + \frac{a(u_1\dot{u}_1 + u_2\dot{u}_2)}{\sqrt{{u_1}^2 + {u_2}^2}}\hat{\mathbf{n}}_3 \tag{274}$$

the required nonholonomic partial accelerations of P in N are readily identified to be

$${}^N\tilde{\mathbf{a}}_1^P = \hat{\mathbf{n}}_1 + \frac{au_1}{\sqrt{{u_1}^2 + {u_2}^2}}\hat{\mathbf{n}}_3, \qquad {}^N\tilde{\mathbf{a}}_2^P = \hat{\mathbf{n}}_2 + \frac{au_2}{\sqrt{{u_1}^2 + {u_2}^2}}\hat{\mathbf{n}}_3 \tag{275}$$

The two equations of interest are then produced from ${}^N\tilde{\mathbf{a}}_r^P \cdot (\mathbf{f} + \mathbf{C} - m\,{}^N\mathbf{a}^P) = {}^N\tilde{\mathbf{a}}_r^P \cdot (\mathbf{f} - m\,{}^N\mathbf{a}^P) = 0$ $(r = 1, 2)$. Although some effort is required because the equations are coupled in $\dot{u}_1$ and $\dot{u}_2$, Eqs. (272) and (273) are recovered.

5.2 A System Containing a Rigid Body

There are certain concepts that the exposition in Sec. 5.1 has in common with that of Ref. [83]. The authors recognize constraint equations that are nonlinear at the velocity level become linear at the acceleration level, and they note the relationship between partial acceleration and partial velocity expressed in Eqs. (201). They make use of these observations to form equations of motion that are equivalent to Eqs. (196), and form generalized constraint forces that are expressed with the final term in Eqs. (207). It is pointed out that the unknown multipliers representing the constraint forces can be eliminated and a reduced set of equations of motion can be obtained. There are, however, a number of differences between what is presented here and in Ref. [83]. In that work, the development is restricted to the case where each motion variable is defined as the time derivative of a generalized coordinate, and remainder terms such as ${}^{N}\mathbf{v}_t^{P_i}$ or ${}^{N}\tilde{\mathbf{v}}_t^{P_i}$ are not accounted for. Constraint forces are not constructed from vector forms of the constraint equations as they are here; therefore an explicit relationship between the multipliers and constraint forces is not provided. As noted previously in Sec. 1.2, their development requires partial velocities to be expressed in a vector basis fixed in an inertial reference frame. The most significant difference is that, although their reduced equations of motion are similar to Eqs. (197), reduction is accomplished by premultiplication with a nonunique orthogonal complement matrix that can be formed in a variety of ways; in contrast, the nonholonomic partial accelerations proposed here are unique once motion variables have been chosen, and they are formed by the same definite process of inspection used to obtain partial velocities. Finally, the Appell-Hamel mechanism is used to illustrate their method, even though it is known that the nonholonomic constraint equations can be expressed in a linear form.

The apparatus of Ref. [83] deals with rigid bodies rather than sets of individual particles; the development of the present approach is completed by fashioning rigid body theorems for the foregoing results.

When particles $P_1, \ldots, P_\beta$ make up a rigid body B, the acceleration ${}^N\mathbf{a}^{P_i}$ in N of a generic particle P_i of B can be written in terms of the acceleration ${}^N\mathbf{a}^{B^\star}$ in N of $B^\star$, the mass center of B, and in terms of the angular acceleration ${}^N\boldsymbol{\alpha}^B$ of B in N

$$ {}^N\mathbf{a}^{P_i} = {}^N\mathbf{a}^{B^\star} + {}^N\boldsymbol{\alpha}^B \times \mathbf{r}_i + {}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r}_i) \qquad (i = 1, \ldots, \beta) \qquad (276) $$

where $\mathbf{r}_i$ is the position vector from $B^\star$ to P_i. Now, ${}^N\boldsymbol{\alpha}^B$ can be expressed uniquely as

$$ {}^N\boldsymbol{\alpha}^B = \sum_{r=1}^{c} {}^N\tilde{\boldsymbol{\alpha}}_r^B \, \dot{u}_r + {}^N\tilde{\boldsymbol{\alpha}}_t^B \qquad (277) $$

where ${}^N\tilde{\boldsymbol{\alpha}}_r^B$ is called the rth *nonholonomic partial angular acceleration* of B in N. Substitution from this relationship and from Eqs. (200) into (276) yields

$$ \begin{aligned} \sum_{r=1}^{c} {}^N\tilde{\mathbf{a}}_r^{P_i} \, \dot{u}_r + {}^N\tilde{\mathbf{a}}_t^{P_i} &= \sum_{r=1}^{c} {}^N\tilde{\mathbf{a}}_r^{B^\star} \, \dot{u}_r + {}^N\tilde{\mathbf{a}}_t^{B^\star} \\ &\quad + \left(\sum_{r=1}^{c} {}^N\tilde{\boldsymbol{\alpha}}_r^B \, \dot{u}_r + {}^N\tilde{\boldsymbol{\alpha}}_t^B \right) \times \mathbf{r}_i + {}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r}_i) \quad (i = 1, \ldots, \beta) \end{aligned} \qquad (278) $$

from which one obtains

$$ {}^N\tilde{\mathbf{a}}_t^{P_i} = {}^N\tilde{\mathbf{a}}_t^{B^\star} + {}^N\tilde{\boldsymbol{\alpha}}_t^B \times \mathbf{r}_i + {}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r}_i) \qquad (i = 1, \ldots, \beta) \qquad (279) $$

and

$$ {}^N\tilde{\mathbf{a}}_r^{P_i} = {}^N\tilde{\mathbf{a}}_r^{B^\star} + {}^N\tilde{\boldsymbol{\alpha}}_r^B \times \mathbf{r}_i \qquad (r = 1, \ldots, c;\ i = 1, \ldots, \beta) \qquad (280) $$

The latter relationship is the nonholonomic partial acceleration analog to nonholonomic partial velocity expressions like Eqs. (4.6.5) and (4.11.16) in Ref. [44] used in the case of simple nonholonomic systems to obtain contributions of B to $\tilde{F}_r$ and $\tilde{F}_r^\star$. Hence, the contribution of B to $\approx{F}_r$ is given by

$$ \begin{aligned} (\overset{\approx}{F}_r)_B &\triangleq \sum_{i=1}^{\beta} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot \mathbf{R}_i \\ &= \sum_{i=1}^{\beta} \left({}^N\tilde{\mathbf{a}}_r^{B^\star} + {}^N\tilde{\boldsymbol{\alpha}}_r^B \times \mathbf{r}_i \right) \cdot \mathbf{R}_i = {}^N\tilde{\mathbf{a}}_r^{B^\star} \cdot \sum_{i=1}^{\beta} \mathbf{R}_i + {}^N\tilde{\boldsymbol{\alpha}}_r^B \cdot \sum_{i=1}^{\beta} \mathbf{r}_i \times \mathbf{R}_i \\ &= {}^N\tilde{\mathbf{a}}_r^{B^\star} \cdot \mathbf{R} + {}^N\tilde{\boldsymbol{\alpha}}_r^B \cdot \mathbf{T} \qquad (r = 1, \ldots, c) \end{aligned} \qquad (281) $$

where the set of all contact and distance forces $\mathbf{R}_i$ acting on the particles of B is equivalent to a force $\mathbf{R}$ whose line of action passes through $B^\star$, together with a couple whose torque is $\mathbf{T}$. The constraint forces and torques that must be applied to B in order to satisfy nonlinear nonholonomic constraint equations may be included in $\mathbf{R}$ and $\mathbf{T}$, or they may be omitted; in either case they will not contribute to $(\tilde{\tilde{F}}_r)_B$. With a similar exercise the contribution of B to $\tilde{\tilde{F}}_r^\star$ is found to be

$$\begin{aligned}
(\tilde{\tilde{F}}_r^\star)_B &\triangleq -\sum_{i=1}^{\beta} {}^N\tilde{\mathbf{a}}_r^{P_i} \cdot m_i\, {}^N\mathbf{a}^{P_i} \\
&= -\sum_{i=1}^{\beta} \left({}^N\tilde{\mathbf{a}}_r^{B^\star} + {}^N\tilde{\boldsymbol{\alpha}}_r^{B} \times \mathbf{r}_i \right) \cdot m_i\, {}^N\mathbf{a}^{P_i} \\
&= -\, {}^N\tilde{\mathbf{a}}_r^{B^\star} \cdot \sum_{i=1}^{\beta} m_i\, {}^N\mathbf{a}^{P_i} - {}^N\tilde{\boldsymbol{\alpha}}_r^{B} \cdot \sum_{i=1}^{\beta} \mathbf{r}_i \times m_i\, {}^N\mathbf{a}^{P_i} \\
&= {}^N\tilde{\mathbf{a}}_r^{B^\star} \cdot \mathbf{R}^\star + {}^N\tilde{\boldsymbol{\alpha}}_r^{B} \cdot \mathbf{T}^\star \qquad (r = 1, \ldots, c)
\end{aligned} \tag{282}$$

where $\mathbf{R}^\star$ and $\mathbf{T}^\star$ are, respectively, the well known inertia force and inertia torque for B in N formed for use with Kane's method.

The procedure for obtaining a minimal set of dynamical equations of motion for a complex nonholonomic system is seen to bear a very close resemblance to Kane's method for simple nonholonomic systems, the only difference being that one uses ${}^N\tilde{\mathbf{a}}_r^{B^\star}$ and ${}^N\tilde{\boldsymbol{\alpha}}_r^{B}$ $(r = 1, \ldots, c)$ rather than ${}^N\tilde{\mathbf{v}}_r^{B^\star}$ and ${}^N\tilde{\boldsymbol{\omega}}_r^{B}$ $(r = 1, \ldots, p)$.

One may be interested in the constraint forces acting on a rigid body, and therefore form equations of motion according to Eqs. (196). In that event it becomes desirable to adapt the process of inspecting a constraint equation written at the acceleration level so that one may identify the direction of a constraint force and the point to which it is applied, together with the direction of a constraint torque and the body upon which it is exerted.

In a constraint equation having the form of (194), the terms associated with $P_1, \ldots, P_\beta$ can be rewritten:

$$\sum_{i=1}^{\beta} {}^N\mathbf{a}^{P_i} \cdot \mathbf{W}_{is} + Z_s$$

$$
\begin{aligned}
&= \sum_{i=1}^{\beta} [\,{}^{N}\mathbf{a}^{Q} + {}^{N}\boldsymbol{\alpha}^{B} \times \mathbf{r}_i + {}^{N}\boldsymbol{\omega}^{B} \times ({}^{N}\boldsymbol{\omega}^{B} \times \mathbf{r}_i)] \cdot \mathbf{W}_{is} + Z_s \\
&= {}^{N}\mathbf{a}^{Q} \cdot \sum_{i=1}^{\beta} \mathbf{W}_{is} + {}^{N}\boldsymbol{\alpha}^{B} \cdot \sum_{i=1}^{\beta} \mathbf{r}_i \times \mathbf{W}_{is} + \sum_{i=1}^{\beta} [\,{}^{N}\boldsymbol{\omega}^{B} \times ({}^{N}\boldsymbol{\omega}^{B} \times \mathbf{r}_i)] \cdot \mathbf{W}_{is} + Z_s \\
&\triangleq {}^{N}\mathbf{a}^{Q} \cdot \mathbf{W}_s + {}^{N}\boldsymbol{\alpha}^{B} \cdot \boldsymbol{\tau}_s + Z'_s \qquad (s = 1, \ldots, l)
\end{aligned} \tag{283}
$$

where $\mathbf{r}_i$ is the position vector from a point Q fixed in B to P_i $(i = 1, \ldots, \beta)$. With the material in Sec. 2.4 in mind, one can therefore inspect a constraint equation written at the acceleration level and conclude that the appearance of the dot product ${}^{N}\mathbf{a}^{Q} \cdot \mathbf{W}_s$ requires that B is subject to a constraint force $\mathbf{C}_s = \lambda_s \mathbf{W}_s$ applied to Q, and the appearance of the dot product ${}^{N}\boldsymbol{\alpha}^{B} \cdot \boldsymbol{\tau}_s$ means B must be acted upon by a couple whose constraint torque is $\mathbf{T}_s = \lambda_s \boldsymbol{\tau}_s$ $(s = 1, \ldots, l)$.

5.3 The Energy Integral

Under certain circumstances there will exist an integral of the equations of motion of a system S in a Newtonian reference frame N (Sec. 7.2, Ref. [44]). An integral, or constant of the motion, is expressed as

$$
\mathcal{H} \triangleq V + K_2 - K_0 = C \tag{284}
$$

where C is a constant, V is a potential energy of S in N, and where K_2 and K_0 are portions of the kinetic energy of S in N that are of degree 2 and 0 respectively in the motion variables. One refers to $\mathcal{H}$ as a Hamiltonian of S in N. Kinetic energy involves squares of velocities in N of the particles belonging to S, suggesting that the energy integral could be regarded as a nonlinear nonholonomic constraint equation. However, when the Hamiltonian is differentiated with respect to time to bring the equation to the acceleration level, it is seen that the result is not independent of the equations of motion and therefore cannot be used to eliminate an equation of motion as has been done previously with nonlinear nonholonomic constraint equations.

The following definition is introduced in the interest of convenience,

$$ {}^N\tilde{\mathbf{u}}^{P_i} \triangleq \sum_{r=1}^{p} {}^N\tilde{\mathbf{v}}_r^{P_i} u_r \qquad (i = 1, \ldots, \nu) \tag{285} $$

so that the velocity of P_i in N is written as

$$ {}^N\mathbf{v}^{P_i} = {}^N\tilde{\mathbf{u}}^{P_i} + {}^N\tilde{\mathbf{v}}_t^{P_i} \tag{286} $$

The portions of kinetic energy appearing in Eq. (284) are then written as [see Eqs. (5.5.9) and (5.5.7) of Ref. [44]]

$$ K_2 = \frac{1}{2}\sum_{i=1}^{\nu} m_i \, {}^N\tilde{\mathbf{u}}^{P_i} \cdot {}^N\tilde{\mathbf{u}}^{P_i}, \qquad K_0 = \frac{1}{2}\sum_{i=1}^{\nu} m_i \, {}^N\tilde{\mathbf{v}}_t^{P_i} \cdot {}^N\tilde{\mathbf{v}}_t^{P_i} \tag{287} $$

The derivatives of K_2 and K_0 with respect to time are needed in what follows and can be written as

$$ \dot{K}_2 = \sum_{i=1}^{\nu} m_i ({}^N\mathbf{a}^{P_i} - \frac{{}^N d}{dt} {}^N\tilde{\mathbf{v}}_t^{P_i}) \cdot ({}^N\mathbf{v}^{P_i} - {}^N\tilde{\mathbf{v}}_t^{P_i}), \qquad \dot{K}_0 = \sum_{i=1}^{\nu} m_i \, {}^N\tilde{\mathbf{v}}_t^{P_i} \cdot \frac{{}^N d}{dt} {}^N\tilde{\mathbf{v}}_t^{P_i} \tag{288} $$

where all vectors appearing in the right hand members of Eqs. (287) are differentiated with respect to time in reference frame N. The energy integral exists if and only if

$$ \sum_{i=1}^{\nu} m_i \, {}^N\mathbf{v}^{P_i} \cdot \frac{{}^N d}{dt} {}^N\tilde{\mathbf{v}}_t^{P_i} = 0 \tag{289} $$

according to Eq. (5.6.1) in Ref. [44].

It is convenient for the moment to regard V as a function of $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_\nu$ and t, where $\mathbf{p}_i$ is the position vector to particle P_i from a point O fixed in N. The time derivative of V is then expressed as

$$ \dot{V} = \sum_{i=1}^{\nu} (\boldsymbol{\nabla}_{\mathbf{p}_i} V) \cdot {}^N\mathbf{v}^{P_i} + \frac{\partial V}{\partial t} = \sum_{i=1}^{\nu} -\mathbf{f}_i \cdot {}^N\mathbf{v}^{P_i} + \frac{\partial V}{\partial t} \tag{290} $$

where $\boldsymbol{\nabla}_{\mathbf{p}_i} V$, the derivative of the scalar V with respect to the position vector $\mathbf{p}_i$, is simply the negative of the force $\mathbf{f}_i$ acting on P_i. An energy integral cannot exist unless $\mathbf{f}_i$ is the only force acting on P_i.

The time derivative of $\mathcal{H}$ can now be assembled.

$$\begin{aligned}\dot{\mathcal{H}} &= \dot{V} + \dot{K}_2 - \dot{K}_0 = 0 \\ &= \sum_{i=1}^{\nu}(m_i\, {}^N\mathbf{a}^{P_i} - \mathbf{f}_i) \cdot ({}^N\mathbf{v}^{P_i} - {}^N\tilde{\mathbf{v}}_t^{P_i}) + \frac{\partial V}{\partial t} - \sum_{i=1}^{\nu}\mathbf{f}_i \cdot {}^N\tilde{\mathbf{v}}_t^{P_i}\end{aligned} \tag{291}$$

The time derivative of the Hamiltonian vanishes when the final two terms vanish,

$$\frac{\partial V}{\partial t} - \sum_{i=1}^{\nu}\mathbf{f}_i \cdot {}^N\tilde{\mathbf{v}}_t^{P_i} = 0 \tag{292}$$

The conditions under which this relationship holds depend on whether S is a holonomic system (in which case ${}^N\tilde{\mathbf{v}}_t^{P_i}$ is replaced by ${}^N\mathbf{v}_t^{P_i}$) or a simple nonholonomic system, and whether the motion variables are the time derivatives of the generalized coordinates or linear combinations thereof. Particular relationships corresponding to the four possibilities are summarized in Table 7.2.1 of Ref. [44]. By regarding V as a function of the generalized coordinates q_1, q_2, ..., q_n and t, one can write

$$\dot{V} = \sum_{r=1}^{n}\frac{\partial V}{\partial q_r}\dot{q}_r + \frac{\partial V}{\partial t} = \frac{\partial V}{\partial t} - \sum_{i=1}^{\nu}\mathbf{f}_i \cdot {}^N\mathbf{v}^{P_i} \tag{293}$$

so that

$$-\sum_{i=1}^{\nu}\mathbf{f}_i \cdot {}^N\mathbf{v}^{P_i} = \sum_{r=1}^{n}\frac{\partial V}{\partial q_r}\dot{q}_r \tag{294}$$

Equations corresponding to each of the four cases are then obtained by employing Eq. (294) to determine an expression for the sum in Eq. (292). For example, when S is a holonomic system and $u_r = \dot{q}_r$ $(r = 1, \ldots, n)$, the velocity of P_i in N is written as ${}^N\mathbf{v}^{P_i} = \sum_{r=1}^{n} {}^N\mathbf{v}_r^{P_i}\dot{q}_r + {}^N\mathbf{v}_t^{P_i}$ and one determines that

$$-\sum_{i=1}^{\nu}\mathbf{f}_i \cdot {}^N\tilde{\mathbf{v}}_t^{P_i} = 0 \tag{295}$$

after equating coefficients of $\dot{q}_r$ in the left and right hand members of Eq. (294). When S is a holonomic system and u_r is a linear combination of the time derivatives of the generalized coordinates so that Eqs. (2.14.5) of Ref. [44] apply, one finds that

$$-\sum_{i=1}^{\nu}\mathbf{f}_i \cdot {}^N\tilde{\mathbf{v}}_t^{P_i} = \sum_{s=1}^{n}\frac{\partial V}{\partial q_s}X_s \tag{296}$$

after equating coefficients of u_r. When S is a simple nonholonomic system similar exercises produce two more relationships, each of which satisfies Eq. (292) and therefore reduces Eq. (291) to

$$\begin{aligned}\dot{\mathcal{H}} = \sum_{i=1}^{\nu}(m_i \, {}^N\mathbf{a}^{P_i} - \mathbf{f}_i) \cdot ({}^N\mathbf{v}^{P_i} - {}^N\tilde{\mathbf{v}}_t^{P_i}) &= \sum_{i=1}^{\nu}(m_i \, {}^N\mathbf{a}^{P_i} - \mathbf{f}_i) \cdot \sum_{r=1}^{p} {}^N\tilde{\mathbf{v}}_r^{P_i} u_r \\ &= -\sum_{r=1}^{p}(\tilde{F}_r^\star + \tilde{F}_r) u_r = 0 \end{aligned} \tag{297}$$

It is now evident that in differentiating the energy integral $\mathcal{H} = C$, the derivative $\dot{\mathcal{H}}$ vanishes precisely because it entails a restatement of the equations of motion. Thus, an independent relationship is not available and an equation of motion cannot be eliminated. In the case of a holonomic system all tildes are removed from Eq. (297) and p is replaced by n.

CHAPTER 6

AN ARGUMENT AGAINST AUGMENTING THE LAGRANGEAN FOR NONHOLONOMIC SYSTEMS

Although it is known that correct dynamical equations of motion for a nonholonomic system cannot be obtained from a Lagrangean that has been augmented with a sum of the nonholonomic constraint equations weighted with multipliers, previous publications suggest otherwise. An example has been proposed in support of augmentation and purportedly demonstrates that an accepted method fails to produce correct equations of motion whereas augmentation leads to correct equations; this chapter shows that in fact the opposite is true. The correct equations, previously discounted on the basis of a flawed application of the Newton-Euler method, are verified by using Kane's method and the new approach to determining the directions of constraint forces. A correct application of the Newton-Euler method reproduces valid equations.

Dealing with nonholonomic constraint equations within the framework of variational methods is a controversial subject. For example, in Ref. [58] Ray modifies Hamilton's principle and augments the Lagrangean by adjoining a sum of nonholonomic constraint equations weighted with multipliers. Later, in Ref. [59], Ray reverses himself. In the erratum he compares the correct way of dealing with constraint equations that are linear in the time derivatives of the generalized coordinates to the incorrect approach of augmenting the Lagrangean that gives the wrong results, even when the constraint equations are linear. Saletan and Cromer follow Ray with Ref. [65], and show the augmented Lagrangean gives correct equations of motion when the constraint equations are holonomic. They conclude that no such augmented Lagrangean exists in the nonholonomic case, in part because they say that there is no way to determine initial conditions needed for the integration of differential equations governing the multipliers. Rosenberg (Ref. [63], p. 220) presents the same demonstration as Ray's erratum and concludes that, although Hamilton's principle may be regarded as a variational principle for conservative holonomic systems, it cannot be so regarded for nonholonomic systems. In an effort to eliminate constraint violations, Rosen and Edelstein make a proposal in Ref. [62] similar to that of Ray 30 years earlier; they account for nonlinear nonholonomic constraint equations in the same way that they do holonomic constraint equations. Hagedorn points out in Ref. [34] that although their approach is justified in the holonomic case, it is incorrect for nonholonomic constraint equations, even when they are linear. He demonstrates this with an example and gives the well-known result for the correct way to handle linear equations, which does not come from modifying the Lagrangean. According to Hagedorn the mistake has been repeated many times over the past century and the pitfall has received attention in Refs. [54], [57], and [82]. More recently, in Ref. [28], Flannery examines the problems encountered by Ray and others and, after in-depth analysis, concludes "General [nonlinear] nonholonomic constraints are completely outside the scope of even the most fundamental principle of D'Alembert. The generalization of any principle based on [D'Alembert's] to general nonholonomic constraints is without foundation."

In their response to his comments, Rosen and Edelstein offer a counterexample purportedly showing that the approach advocated by Hagedorn leads to incorrect results. Their conclusion is based on a flawed application of the Newton-Euler method; the mistake by the former authors, as well as the validity of the approach taken by the latter author, both become readily apparent by treating the problem with the method developed in Chapter 2.

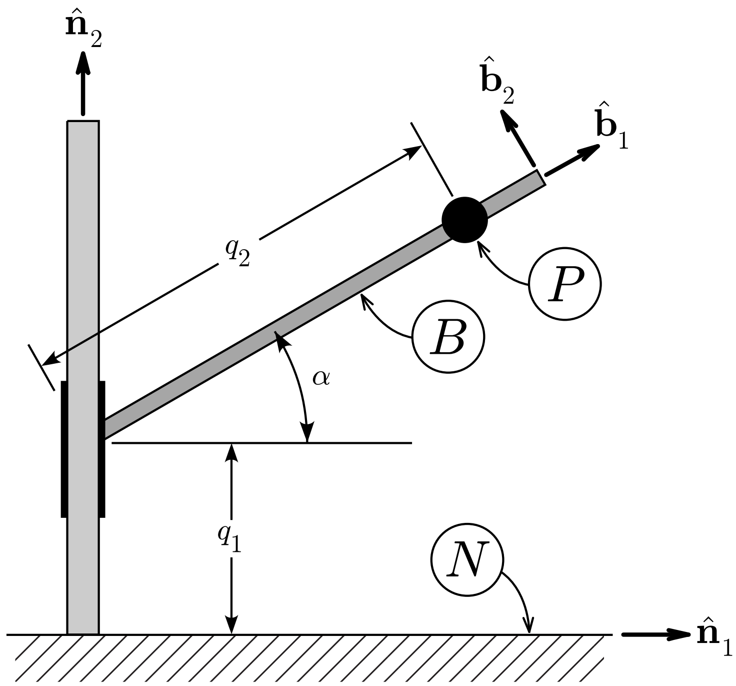


Figure 24: A Particle Moving on a Sliding Inclined Rod

The planar system featured in the counterexample is shown in Figure 24. Two perpendicular unit vectors $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ are fixed in an inertial reference frame N. A smooth rod B whose axis is parallel to unit vector $\hat{\mathbf{b}}_1$ is inclined at a constant angle α to $\hat{\mathbf{n}}_1$; B is permitted to translate along an axis parallel to $\hat{\mathbf{n}}_2$. A particle P of mass m moves along B, and the mass of B is negligible in comparison to m. It is said that no forces are exerted on P other than those necessary to prevent it from losing contact with B.

Analysis is facilitated by working with two generalized coordinates q_1 and q_2 shown in Figure 24, where q_1 is the displacement in a prismatic joint connecting B to N, and where q_2 is the displacement of P along the rod. Two motion variables are introduced simply as $u_r = \dot{q}_r$ $(r = 1, 2)$. A motion constraint is to be imposed upon the velocity of P in B, expressed by the relationship

$$\varepsilon \cos\alpha\, u_2 - q_1 = 0 \tag{298}$$

where ε is a positive constant. The equivalence of this expression and the constraint equation in Ref. [34] is demonstrated presently. Now, the velocity ${}^N\mathbf{v}^{\overline{B}}$ in N of every point $\overline{B}$ fixed in B is given by ${}^N\mathbf{v}^{\overline{B}} = u_1\hat{\mathbf{n}}_2$, and the velocity ${}^B\mathbf{v}^P$ of P in B is given by ${}^B\mathbf{v}^P = u_2\hat{\mathbf{b}}_1$. Henceforth, $\overline{B}$ is taken to be the point of B that is coincident with P, and the velocity of P in N is simply ${}^N\mathbf{v}^P = {}^N\mathbf{v}^{\overline{B}} + {}^B\mathbf{v}^P$. The nonholonomic constraint equation (298) can thus be written in vector form as

$$({}^N\mathbf{v}^P - {}^N\mathbf{v}^{\overline{B}}) \cdot \hat{\mathbf{b}}_1 - \frac{q_1}{\varepsilon \cos\alpha} = 0 \tag{299}$$

Inspection of this equation indicates that P must be subject to a constraint force $\mathbf{C}_1$ that is parallel to $\hat{\mathbf{b}}_1$,

$$\mathbf{C}_1 = \mu_1\hat{\mathbf{b}}_1 \tag{300}$$

and a force $-\mathbf{C}_1$ is applied to B at $\overline{B}$. In practice this set of forces could be applied with a motorized gear attached to P moving on a track of gear teeth fixed in B. Alternatively, friction could be exploited by using a capstan and pinch roller on opposite sides of B in the way a similar mechanism is used to transport magnetic tape. Evidently the rod cannot be perfectly smooth as hypothesized in the problem statement, if the nonholonomic constraint equation (298) is to be obeyed. There also exists a configuration constraint that prevents P from moving in B in the direction of $\hat{\mathbf{b}}_2$; at the velocity level, the holonomic constraint equation is expressed as

$$({}^N\mathbf{v}^P - {}^N\mathbf{v}^{\overline{B}}) \cdot \hat{\mathbf{b}}_2 = 0 \tag{301}$$

and inspection of this relationship reveals that P is acted upon by the constraint force

$$\mathbf{C}_2 = \mu_2\hat{\mathbf{b}}_2 \tag{302}$$

whereas the constraint force applied to $\overline{B}$ is $-\mathbf{C}_2$. (In Ref. [34] the magnitude of the reaction force $\mathbf{C}_2$ normal to $\hat{\mathbf{b}}_1$ is denoted by N rather than μ_2.)

After forming the acceleration of P in N as ${}^N\mathbf{a}^P = \dot{u}_1\hat{\mathbf{n}}_2 + \dot{u}_2\hat{\mathbf{b}}_1$, one is in a position to form two equations of motion for the system S composed of P and B,

$F_r + F_r^\star = 0$ $(r = 1, 2)$. The holonomic generalized active forces are given by

$$F_r = {}^N\mathbf{v}_r^P \cdot (\mathbf{C}_1 + \mathbf{C}_2) + {}^N\mathbf{v}_r^{\overline{B}} \cdot (-\mathbf{C}_1 - \mathbf{C}_2) \qquad (r = 1, 2) \tag{303}$$

and the holonomic generalized inertia forces $F_r^\star$ are constructed according to

$$F_r^\star = -\,{}^N\mathbf{v}_r^P \cdot m\,{}^N\mathbf{a}^P \qquad (r = 1, 2) \tag{304}$$

The required holonomic partial velocities are

$${}^N\mathbf{v}_1^P = \hat{\mathbf{n}}_2, \qquad {}^N\mathbf{v}_2^P = \hat{\mathbf{b}}_1, \qquad {}^N\mathbf{v}_1^{\overline{B}} = \hat{\mathbf{n}}_2, \qquad {}^N\mathbf{v}_2^{\overline{B}} = \mathbf{0} \tag{305}$$

so that the dynamical equations of motion for P are found to be

$$m(\dot{u}_1 + \sin\alpha\,\dot{u}_2) = 0 \tag{306}$$

$$m(\sin\alpha\,\dot{u}_1 + \dot{u}_2) = \mu_1 \tag{307}$$

The nonholonomic constraint force $\mathbf{C}_1$ contributes to the holonomic generalized active forces, whereas the holonomic constraint force $\mathbf{C}_2$ does not. A third equation is needed to solve for the three unknowns $\dot{u}_1$, $\dot{u}_2$, and μ_1; it is provided by the nonholonomic constraint equation (298) expressed at the acceleration level,

$$\dot{u}_2 - \frac{u_1}{\varepsilon\,\cos\alpha} = 0 \tag{308}$$

An analytical solution is then available,

$$\dot{u}_1 = -\frac{\tan\alpha}{\varepsilon}u_1 \tag{309}$$

$$\dot{u}_2 = \frac{u_1}{\varepsilon\,\cos\alpha} \tag{310}$$

$$\mu_1 = \frac{m\cos\alpha}{\varepsilon}u_1 \tag{311}$$

It is worth noting that Eqs. (300), (310), and (311) together contradict the statement in Ref. [34] preceding Eq. (16) therein. The acceleration of P along the path (the rod) is in general nonzero if the proposed nonholonomic constraint equation is to be satisfied; it vanishes only in the special case when the rod is stationary ($u_1 = 0$).

The differential equation (309) yields a closed-form solution

$$u_1 = K_1 e^{-\rho t} \tag{312}$$

where $\rho \triangleq (\tan\alpha)/\varepsilon$ as defined in Ref. [34], and the constant of integration K_1 is the value of u_1 at $t = 0$. This solution in turn facilitates integration of Eq. (310), yielding

$$u_2 = -\frac{K_1}{\sin\alpha} e^{-\rho t} + K_2 \tag{313}$$

where K_2 is determined once the value of u_2 at $t = 0$ is specified. Integration of the two kinematical differential equations $\dot{q}_r = u_r$ $(r = 1, 2)$ produces solutions for the generalized coordinates.

$$q_1 = -\frac{K_1}{\rho} e^{-\rho t} + K_3, \qquad q_2 = \frac{K_1}{\rho \sin\alpha} e^{-\rho t} + K_2 t + K_4 \tag{314}$$

where the constants of integration K_3 and K_4 can be evaluated on the basis of the initial conditions of q_1 and q_2. It must be noted that the initial values of u_2 and q_1 have to satisfy Eq. (298); a similar recognition appears in Ref. [34].

One is now in a position to show that Eqs. (312)–(314) verify the results attributed in Ref. [34] to Hagedorn's approach. First, relationships between the Cartesian coordinates x and y and the generalized coordinates q_1 and q_2 are established.

$$x = \cos\alpha\, q_2, \qquad \dot{x} = \cos\alpha\, u_2, \qquad \ddot{x} = \cos\alpha\, \dot{u}_2 \tag{315}$$

$$y = q_1 + \sin\alpha\, q_2, \qquad \dot{y} = u_1 + \sin\alpha\, u_2, \qquad \ddot{y} = \dot{u}_1 + \sin\alpha\, \dot{u}_2 \tag{316}$$

Appropriate substitution from these relationships shows that the original form of the nonholonomic constraint equation given in Ref. [34], $y - x\tan\alpha - \varepsilon\dot{x} = 0$, gives way to Eq. (298). Furthermore, Eqs. (12) and (14a) in Sec. 3 of Ref. [34] are recovered from Eqs. (314) here.

$$x = \cos\alpha\, q_2 = \frac{K_1}{\rho\tan\alpha} e^{-\rho t} + K_2 \cos\alpha\, t + K_4 \cos\alpha \triangleq D_3 e^{-\rho t} + D_4\, t + D_5 \tag{317}$$

$$y = q_1 + \sin\alpha\, q_2 = -\frac{K_1}{\rho} e^{-\rho t} + K_3 + \frac{K_1}{\rho} e^{-\rho t} + K_2 \sin\alpha\, t + K_4 \sin\alpha \triangleq D_1\, t + D_2 \tag{318}$$

It is clear that because of the constraint described by Eq. (298), only three of the constants of integration K_1, K_2, K_3, and K_4 are independent, and only three of the five constants $D_1, \ldots, D_5$ used in Ref. [34] are independent.

The second-order differential equations (11a) and (11b) in Sec. 3 of Ref. [34] can also be recovered from Eqs. (309) – (311) here. Dealing with the differential equation for y is straightforward,

$$\ddot{y} = \dot{u}_1 + \sin\alpha\,\dot{u}_2 = -\frac{\tan\alpha}{\varepsilon}u_1 + \sin\alpha\frac{u_1}{\varepsilon\,\cos\alpha} = 0 \tag{319}$$

The differential equation for x can be rewritten as

$$\ddot{x} = \cos\alpha\,\dot{u}_2 = \cos\alpha\frac{u_1}{\varepsilon\,\cos\alpha} = \frac{\mu_1}{m\cos\alpha} \tag{320}$$

A relationship between the multiplier μ_1 used here and the multiplier λ used in Ref. [34] is required, and can be obtained by rewriting the original nonholonomic constraint equation as

$$-\,\varepsilon\dot{x} + y - x\tan\alpha = {}^{N}\mathbf{v}^{P}\cdot(-\varepsilon\hat{\mathbf{n}}_1) + y - x\tan\alpha = 0 \tag{321}$$

Inspection of this equation indicates that a constraint force parallel to the vector $-\varepsilon\hat{\mathbf{n}}_1$ must be applied to P. Consequently,

$$\mathbf{C}_1' = -\lambda\varepsilon\hat{\mathbf{n}}_1 \tag{322}$$

The projection of $\mathbf{C}_1'$ onto $\hat{\mathbf{b}}_1$ must be the same as that of $\mathbf{C}_1$; therefore,

$$\mathbf{C}_1\cdot\hat{\mathbf{b}}_1 = \mathbf{C}_1'\cdot\hat{\mathbf{b}}_1 = \mu_1 = -\lambda\varepsilon\cos\alpha \tag{323}$$

Hence, Eq. (320) is rewritten

$$\ddot{x} + \frac{\lambda\varepsilon}{m} = 0 \tag{324}$$

in agreement with Eq. (11a) of Ref. [34] when m is taken as unity.

Rosen and Edelstein reject the preceding differential equations for x and y, and the closed-form solutions, on the basis of their results obtained with the Newton-Euler method. With the analysis already performed here it is evident that their application of the method is flawed, and the point in their development where the mistake was made can be identified immediately. In what follows, a correct application of the Newton-Euler approach is shown to yield the foregoing results.

In their Eqs. (2a) and (2b), Rosen and Edelstein do not account for the constraint force $\mathbf{C}_1'$ needed to ensure satisfaction of their nonholonomic constraint equation; they only consider $\mathbf{C}_2$ required to bring about the configuration constraint. Upon writing $\mathbf{C}_1' + \mathbf{C}_2 = m\,^N\mathbf{a}^P$, it is seen that Eqs. (2) should be stated

$$m\ddot{x} = -N \sin\alpha - \lambda\varepsilon, \qquad m\ddot{y} = N\cos\alpha \tag{325}$$

or

$$m\cos\alpha\,\dot{u}_2 = -N\sin\alpha + \mu_1/\cos\alpha, \qquad m(\dot{u}_1 + \sin\alpha\,\dot{u}_2) = N\cos\alpha \tag{326}$$

Application of Newton's second law to P must be accompanied by its application to B. In addition to the reaction forces $-\mathbf{C}_1'$ and $-\mathbf{C}_2$ acting at $\overline{B}$, a reaction force $\mu_3\hat{\mathbf{n}}_1$ is applied at the prismatic joint, therefore we write $\mu_3\hat{\mathbf{n}}_1 - \mathbf{C}_1' - \mathbf{C}_2 = m_B\,^N\mathbf{a}^{\overline{B}}$; however, the mass of B is neglected in comparison to m so $\mu_3\hat{\mathbf{n}}_1 - \mathbf{C}_1' - \mathbf{C}_2 = \mathbf{0}$. That is,

$$\mu_3 + \lambda\varepsilon + N\sin\alpha = 0, \qquad -N\cos\alpha = 0 \tag{327}$$

The first of these can be used if μ_3 is of interest but the relationship is not important in what remains to be done. The second of these reveals that $N = 0$ (so $\mu_2 = 0$) and, as an immediate consequence, $\ddot{y} = 0$, in agreement with what has been previously shown. Equations (326) can now be simplified, and the nonholonomic constraint equation (308) at the acceleration level is again brought to bear so that

$$m\cos\alpha\,\dot{u}_2 - \frac{\mu_1}{\cos\alpha} = 0, \qquad \dot{u}_1 + \sin\alpha\,\dot{u}_2 = 0, \qquad \dot{u}_2 - \frac{u_1}{\varepsilon\,\cos\alpha} = 0 \tag{328}$$

constitute three equations in three unknowns, $\dot{u}_1$, $\dot{u}_2$, and μ_1. They lead immediately to Eqs. (309)–(311).

A straightforward application of Kane's method for simple nonholonomic systems, together with identification of the constraint forces needed to impose a motion constraint and a configuration constraint, are used to verify results obtained with what is called the regular variational approach, brought to the reader's attention by Hagedorn in Ref. [34]. Further, the rationale used to reject the associated results is shown to be defective. The conclusion by Hagedorn, Ray, Flannery, and others is thus affirmed; namely, the Lagrangean cannot be augmented by the sum of nonholonomic constraint equations weighted with multipliers, regardless of whether or not such equations are linear in the time derivatives of the generalized coordinates.

CHAPTER 7

CONCLUSIONS

This thesis sets forth a method for identifying a set of forces required to constrain the behavior of any mechanical system modeled as a set of particles and rigid bodies. The method can be applied whenever a constraint equation can be expressed at the acceleration level in terms of dot products such as ${}^{N}\mathbf{a}^{P} \cdot \mathbf{W}$, where ${}^{N}\mathbf{a}^{P}$ is the acceleration of a point P in a Newtonian reference frame N. The appearance of such a dot product indicates a constraint force $\lambda\mathbf{W}$ must be applied to P, where λ is a scalar multiplier. A constraint equation that is nonlinear in acceleration and thus contains a term such as ${}^{N}\mathbf{a}^{P} \cdot {}^{N}\mathbf{a}^{P}$ cannot be treated with this method. If a constraint can be expressed at the velocity level in terms of dot products of the form ${}^{N}\mathbf{v}^{P} \cdot \mathbf{W}$, then a requirement for a constraint force $\lambda\mathbf{W}$ is deduced by inspecting the constraint equation at the velocity level instead of at the acceleration level. Here, ${}^{N}\mathbf{v}^{P}$ denotes the velocity of P in N and the constraint equation is said to be linear in velocity. All configuration constraints, and a broad class of classical motion constraints, can be expressed at the velocity level with relationships that are linear in velocity. Motion constraint equations that are nonlinear in velocity, and represent nonclassical servo-constraints or program constraints, become linear in acceleration after they are differentiated with respect to time in N.

With regard to configuration constraints and motion constraints described by expressions that are linear in velocity, it is shown that constraint forces obtained by the foregoing technique of inspection are in fact nonworking in the case of Lagrange's equations for holonomic systems, and noncontributing when one forms minimal sets of Kane's equations. If this were not so, the method would be highly suspect. Furthermore, it is demonstrated that the multipliers introduced here can be identical to measure numbers of constraint forces brought into evidence by Kane's procedure.

When a mechanical system contains a rigid body B, the angular acceleration ${}^N\boldsymbol{\alpha}^B$ of B in N may appear in a dot product such as ${}^N\boldsymbol{\alpha}^B \cdot \boldsymbol{\tau}$ in a constraint equation at the acceleration level, in which case a constraint couple whose torque is $\lambda\boldsymbol{\tau}$ must be exerted on B. The necessity for this constraint torque can be determined by inspecting a constraint equation at the velocity level if it contains the dot product ${}^N\boldsymbol{\omega}^B \cdot \boldsymbol{\tau}$, where ${}^N\boldsymbol{\omega}^B$ is the angular velocity of B in N.

The method is free of deficiencies present in existing approaches, including the lack of generality, comprehensiveness, consistency, and conciseness. Some of the methods in current use entail wasted effort and lead to results that are at odds with physical reasoning. Others cannot be applied in a selective manner to examine only those particular constraints that are of interest; it's all or nothing. Still others unnecessarily limit the physical meaning of the multipliers or, what is worse, provide no clear correspondence between the multipliers and the actual constraint forces. The predominant use of scalars leads to many complications that can be avoided by using basis-independent vectors (as distinguished from matrix analogs or scalar representations of vectors) whenever possible. Vectors are used to great advantage in the method proposed here; pitfalls associated with scalars are avoided, and a rigorous justification of the method via Newton's second law is facilitated. The technique of inspection systematically establishes the directions of constraint forces very soon after a constraint equation is available in vector form, generally much sooner and with less labor than when working with constraint equations written entirely in terms of scalars. The method is especially advantageous in cases such as those discussed in Chapters 5 and 6 where the required direction of a constraint force is not otherwise obvious. To a certain extent the vectors $\mathbf{W}$ and $\boldsymbol{\tau}$ can be chosen at the convenience of the analyst.

The method is manifestly comprehensive as it is applicable to a wide range of constraints encountered in practice, including those associated with confinement of a particle in a rigid body, joints, prescribed position, constant distance between particles, impenetrability of rigid bodies, rolling, sharp-edged blades, and Kepler's laws of planetary motion. It is applicable also to less commonly considered servo-constraints such as the requirement that the velocities in N of two particles remain parallel, equal in magnitude, or perpendicular, and the restriction on one particle's velocity proposed by Appell. The method plays a central role in discovering the flaws in an assertion made in support of an incorrect approach to dealing with nonholonomic constraint equations within the framework of variational principles.

The use of partial accelerations, instead of the partial velocities normally employed with Kane's method, leads to the development of two new approaches for deriving equations of motion for a complex nonholonomic system, namely one subject to constraints expressed at the velocity level with equations that are nonlinear in velocity. The two algorithms enable construction of dynamical equations that either do or do not contain evidence of the constraint forces, according to the interests of the analyst.

Continued investigation is warranted in some areas of this research. First, the matter of choosing additional motion variables deserves further study. Should they be introduced as recommended in Refs. [52] and [10], even though doing so complicates a solution for constraint force measure numbers? Must the question be answered on a case by case basis, or can rules of thumb be formulated? Second, it would be instructive to apply the results obtained in Sec. 5.2 to problems in which a complex nonholonomic system contains rigid bodies. Finally, possible benefits of using the method proposed here in symbol manipulation programs such as AUTOLEV should be explored. The process of inspecting an equation to identify vector coefficients is already performed by AUTOLEV in order to obtain partial velocities and partial angular velocities; therefore, identifying directions of constraint forces and torques by examining constraint equations would seem to be a naturally suitable task. Incorporation of the method proposed here might simplify the derivation of equations of motion for constrained systems, or the way in which a user interacts with the program, or both. Additionally, the use of partial accelerations and partial angular accelerations to deal with complex nonholonomic systems should prove to be straightforward.

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